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MATHÉMATIQUES

SUR QUELQUES MODIFICATIONS DE L'INÉGALITÉ DE TCHEBYCHEFF

Par S. BERNSTEIN, de l'Académie

Soient $z_1, z_2, \ldots, z_n$ une suite de grandeurs aléatoires telles que l'espérance mathématique conditionnelle $\mathfrak{M}_{(i)}(z_{i+1})=0$, quelles que soient les valeurs des grandeurs antérieures $z_1, z_2, \ldots, z_i$. Dans ces conditions la dispersion de la somme $\sum_{i=1}^{n} z_i$ est égale à $B_n = \sum_{i=1}^{n} \beta_i$, où $\beta_i = \mathfrak{M} z_i^2$.

On peut démontrer les propositions suivantes au sujet des sommes de ces grandeurs z_i.

Théorème 1. *La probabilité que toutes les inégalités*

$$|z_1 + z_2 + \ldots + z_i| \leqslant t\sqrt{B_n} \qquad (i = 1, 2, \ldots, n), \tag{1}$$

seront réalisées simultanément est supérieure à $1 - \frac{1}{t^2}$.

Je ne m'arrêterai pas sur la démonstration de ce théorème présentant une généralisation directe de l'inégalité classique de Tchebycheff, car celle-ci est plus simple et entièrement analogue à la démonstration du théorème suivant qui correspond à une des modifications de l'inégalité mentionnée que j'ai donnée autrefois.

Théorème II. *Soit* $\beta_{k,\,k-1} = \mathfrak{M}_{(k-1)} z_k^2$ *la dispersion conditionnelle de* z_k, *lorsque* $z_1, z_2, \ldots, z_{k-1}$ *sont données; soit*

$$\frac{\beta_{k,\,k-1}}{\beta_k} \leqslant R_k \,. \tag{2}$$

S'il existe un nombre H tel que

$$\mathfrak{M}_{(k-1)}\, z_k^l \leqslant \frac{\beta_{k,\,k-1}}{2} H^{l-2}\, l\,! \tag{3}$$

la probabilité P de la réalisation simultanée de toutes les inégalités

$$z_1 + z_2 + \ldots + z_i < 2t\sqrt{B_n'} \tag{4}$$

est supérieure à $1 - e^{-t^2}$, pourvu que $0 < t \leqslant \frac{\sqrt{B_n'}}{2H}$, où $B_n' = \sum_{i=1}^{n} R_i \beta_i$.

En effet, soit $\varepsilon > 0$ et soit Q_i la probabilité de la réalisation simultanée de l'inégalité

$$e^{\varepsilon(z_1+z_2+\ldots+z_i)} \geqslant L \tag{5}$$

avec les inégalités

$$e^{\varepsilon(z_1+\ldots+z_k)} < L \qquad (k<i), \tag{6}$$

de sorte que

$$Q = \sum_{i=1}^{n} Q_i \tag{7}$$

représente la probabilité qu'il y aura au moins une valeur $i \leqslant n$ pour laquelle l'inégalité (5) sera réalisée.

Soit

$$I = \mathfrak{M}\left(e^{\varepsilon(z_1+z_2+\ldots+z_n)}\right) \tag{8}$$

et soient $z_1^{(i)}, z_2^{(i)}, \ldots, z_i^{(i)}$ une certaine succession de valeurs de $z_1, z_2 \ldots, z_i$ satisfaisant aux inégalités (5) et (6); dans ces conditions les termes correspondants de I peuvent être mis sous la forme

$$p_i\, e^{\varepsilon(z_1^{(i)}+z_2^{(i)}+\ldots+z_i^{(i)})}\, \mathfrak{M}_{(i)}\, e^{\varepsilon(z_{i+1}+\ldots+z_n)},$$

où p_i est la probabilité des égalités $z_1 = z_1^{(i)}$, $z_2 = z_1^{(i)}$, ..., $z_i = z_i^{(i)}$ tandis que l'espérance mathématique conditionnelle

$$\mathfrak{M}_{(i)}\, e^{\varepsilon(z_{i+1}+\ldots+z_n)} \geqslant 1$$

à cause de l'inégalité évidente

$$\sum P_k\, e^{\sigma_k} \geqslant 1$$

qui a lieu toutes les fois que $\sum P_k = 1$, $\sum P_k \sigma_k = 0$, $P_k \geqslant 0$.

Donc en faisant la somme $I^{(i)}$ de tous ces termes, en remarquant que $\sum p_i = Q_i$, nous avons à cause de (5):

$$I^{(i)} \geqslant LQ_i,$$

d'où

$$I \geqslant \sum_{i=1}^{n} I^{(i)} \geqslant L \sum_{i=1}^{n} Q_i = LQ,$$

ou bien

$$Q \leqslant \frac{I}{L}. \tag{9}$$

D'autre part, on a

$$\mathfrak{M}_{(i-1)}\, e^{\varepsilon z_i} = 1 + \frac{\varepsilon^2}{2}\,\beta_{i,\,i-1} + \sum_{l=3}^{\infty} \frac{\varepsilon^l}{l!}\,\mathfrak{M}_{(i-1)}\, z_i^l$$

et en tenant compte de (3)

$$\mathfrak{M}_{(i-1)}\, e^{\varepsilon z_i} \leqslant 1 + \beta_{i,\,i-1} \sum_{l=2}^{\infty} \frac{\varepsilon^2}{2}\,(\varepsilon H)^{l-2} \leqslant 1 + \varepsilon^2\,\beta_{i,\,i-1} < e^{\varepsilon^2 \beta_{i,\,i-1}}, \qquad (10)$$

en supposant que

$$\varepsilon H \leqslant \frac{1}{2}. \qquad (11)$$

Ainsi, on a à cause de (2)

$$\mathfrak{M}_{(i-1)}\, e^{\varepsilon z_i} < e^{\varepsilon^2 R_i \beta_i},$$

d'où il résulte que

$$I < e^{\varepsilon^2 \sum_{i=1}^{n} R_i \beta_i} = e^{\varepsilon^2 B'_n}. \qquad (12)$$

Par conséquent, nous tirons de (9) que

$$Q < \frac{e^{\varepsilon^2 B'_n}}{L}, \qquad (13)$$

où Q est la probabilité qu'il y aura au moins une valeur de $i \leqslant n$ pour laquelle

$$\varepsilon(z_1 + z_2 + \ldots + z_i) \geqslant \log L. \qquad (14)$$

Or, si l'on pose

$$\log L = t^2 + \varepsilon^2 B'_n,$$

les inégalités (14) prennent la forme:

$$z_1 + z_2 + \ldots + z_i \geqslant \frac{t^2}{\varepsilon} + \varepsilon B'_n, \qquad (15)$$

et la probabilité correspondante Q satisfait à l'inégalité

$$Q < e^{-t^2}. \qquad (16)$$

En particulier, en posant $\varepsilon = \frac{t}{\sqrt{B'_n}}$ [il suffit pour satisfaire à (11) de prendre $t \leqslant \frac{\sqrt{B'_n}}{2H}$] nous arrivons à l'affirmation du théorème.

Observons que le théorème II conduit immédiatement à l'extension de «la loi du logarithme itéré» (pour $B_n \to \infty$) aux grandeurs aléatoires dépendantes qui y figurent; l'énoncé correspondant de «la loi» ne sera pas modifié, si $R_n \to 1$ pour $n = \infty$.

On transforme d'une façon analogue les autres modifications de l'inégalité de Tchebycheff qui se trouvent dans mon article* „Sur une modification de l'inégalité de Tchebycheff et sur l'évaluation de l'approximation de la formule de Laplace" et dans mon cours du calcul des probabilités.

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MATHÉMATIQUES

SUR QUELQUES APPROXIMATIONS DANS LE PROBLÈME DES MOMENTS POUR LES FONCTIONS DE DEUX VARIABLES

Par M. KRAVČUK, de l'Académie des Sciences de la RSS d'Ukraine

Dans la note précédente* nous avons démontré le théorème suivant. Soient $P(x)$ et $Q(x)$ deux fonctions non négatives et non décroissantes dans l'intervalle $(\alpha, \alpha + S)$. Alors on a l'inégalité:

$$\left| \int_{\alpha}^{x} dQ(x) - \int_{\alpha}^{x} dP(x) \right| < \left(\frac{SM[p]}{m} + \zeta \right) (A + B \log n) \qquad (1)$$

$$(\alpha \leqslant x \leqslant \alpha + S),$$

où

$$p = \frac{dP}{dx},$$

$$M[p] = \limsup |p(x)| \qquad (\alpha \leqslant x \leqslant \alpha + S),$$

$$\zeta \geqslant |2\zeta_k|,$$

$$\zeta_k = \int_{\alpha}^{\alpha+S} \cos k\pi \frac{x-\alpha}{S} dQ(x) - \int_{\alpha}^{\alpha+S} \cos k\pi \frac{x-\alpha}{S} dP(x)$$

$$(k = 0, 1, \ldots, m),$$

A et B étant deux constantes absolues.

Dans cette note on établit un résultat analogue pour les fonctions de deux variables. Soient $P(x, y)$, $Q(x, y)$ deux fonctions non négatives et satisfaisant aux conditions:

$$\Delta^2_{xy} P \geqslant 0, \quad \Delta^2_{xy} Q \geqslant 0,$$

dans le rectangle aux sommets

$$(\alpha, \beta), \quad (\alpha + S, \beta), \quad (\alpha + S, \beta + T), \quad (\alpha, \beta + T). \qquad (2)$$

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* C. R. Acad. Sci. URSS, XIV, № 3, 91 (1937). Il est nécessaire de faire dans la note citée les corrections suivantes: 1) page 93, lignes 8—27 sont à omettre, comme inutiles et incorrectes dans le cas général, 2) page 95, ligne 6, lire $0 < \vartheta < 1$ au lieu de $0 < \vartheta \leqslant 1$.

Alors, en introduisant la notation:

$$\varepsilon_{kl} = \int_{\alpha}^{\alpha+S} \int_{\beta}^{\beta+T} \cos k\pi \frac{x-\alpha}{S} \cos l\pi \frac{y-\beta}{T} (d^2_{xy} Q - d^2_{xy} P) \tag{3}$$

$$(k = 0,\ 1,\ \ldots\ ,\ m)$$
$$(l = 0,\ 1,\ \ldots\ ,\ n),$$

on a l'inégalité:

$$\left| \int_{\alpha}^{x} \int_{\beta}^{y} d^2_{xy} Q - \int_{\alpha}^{x} \int_{\beta}^{y} d^2_{xy} P \right| \leqslant$$

$$\leqslant \left\{ \frac{S}{m} M\left[\frac{\partial P}{\partial x}\right] + \frac{T}{n} M\left[\frac{\partial P}{\partial y}\right] + \varepsilon \right\} \cdot (A + B\log m)(A + B\log n) \tag{4}$$

$$(\alpha \leqslant x \leqslant \alpha + S,\quad \beta \leqslant y \leqslant \beta + T),$$

où A, B sont deux constantes absolues, ε est un nombre quelconque satisfaisant à la condition:

$$\varepsilon \geqslant |4\varepsilon_{kl}|,$$

et où l'on désigne par $M\left[\frac{\partial P}{\partial x}\right]$, $M\left[\frac{\partial P}{\partial y}\right]$ respectivement les limites supérieures des fonctions $\left|\frac{\partial P}{\partial x}\right|$, $\left|\frac{\partial P}{\partial y}\right|$ dans le rectangle (2).

Pour démontrer le résultat (4), considérons les fonctions:

$$\left.\begin{aligned} Q_l(x) &= \int_{\beta}^{\beta+T} \left(1 + \cos l\pi \frac{y-\beta}{T}\right) \int_{\alpha}^{x} d^2_{xy} Q \\ P^2_l(x) &= \int_{\beta}^{\beta+T} \left(1 + \cos l\pi \frac{y-\beta}{T}\right) \int_{\alpha}^{x} d^2_{xy} P, \end{aligned}\right\} \tag{5}$$

non négatives et non décroissantes dans l'intervalle $(\alpha,\ \alpha + S)$.

Alors les égalités (3) donnent:

$$\int_{\alpha}^{\alpha+S} \cos k\pi \frac{x-\alpha}{S} \left\{ dQ_l(x) - dP_l(x) \right\} = \varepsilon_{kl} + \varepsilon_{k0}$$
$$(k = 0,\ 1,\ \ldots\ ,\ m).$$

Il en résulte pour la fonction

$$\eta_l(x) = \int_{\alpha}^{x} dQ_l(x) - \int_{\alpha}^{x} dP_l(x) \tag{6}$$

l'inégalité suivante:

$$|\eta_l(x)| < \left(\frac{SM\left[\frac{\partial P_l}{\partial x}\right]}{m} + \varepsilon_l \right) (A + B\log m), \tag{7}$$

comme une application immédiate de la formule (1).

La formule (7) nous conduit aux égalités:

$$\int_{\beta}^{\beta+T} \cos l\pi \frac{y-\beta}{T} \left(\int_{\alpha}^{x} d^2_{xy}Q - \int_{\alpha}^{x} d^2_{xy}P \right) = \eta_l(x) + \frac{1}{2}\eta_0(x) \qquad (8)$$

$$(l = 0,\ 1,\ \ldots,\ n),$$

d'où, à l'aide de la même inégalité (1), on tire:

$$\left| \int_{\alpha}^{x}\int_{\beta}^{y} d^2_{xy}Q - \int_{\alpha}^{x}\int_{\beta}^{y} d^2_{xy}P \right| \leqslant \left\{ \frac{T}{n} M\left[\frac{\partial P}{\partial y}\right] + \eta \right\} \cdot (A + B\log n), \qquad (9)$$

η étant un nombre quelconque, satisfaisant à la condition

$$\eta \geqslant |2\eta_l(x) + \eta_0(x)| \quad (l = 0,\ 1,\ \ldots,\ n).$$

Les inégalités (9) et (7) donnent, comme conséquence immédiate, la formule (4).

Dans le cas particulier, où tous les nombres ε_{kl} sont zéros (problème des moments classique), l'inégalité (4) se réduit à la suivante:

$$\left| \int_{\alpha}^{x}\int_{\beta}^{y} d^2_{xy}Q - \int_{\alpha}^{x}\int_{\beta}^{y} d^2_{xy}P \right| \leqslant$$

$$\leqslant \left\{ \frac{S}{m} M\left[\frac{\partial P}{\partial x}\right] + \frac{T}{n} M\left[\frac{\partial P}{\partial y}\right] \right\} \cdot (A + B\log m)\,(A + B\log n).$$

On peut établir sans peine un résultat analogue à l'inégalité (9′) de la note citée.

La généralisation pour le cas d'un nombre quelconque de variables est immédiate.

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MATHEMATIK

NEUE VERALLGEMEINERUNG DER FASTPERIODISCHEN FUNKTIONEN

Von B. LEWITAN

(Vorgelegt von S. N. Bernstein, Mitglied der Akademie, d. 2. X. 1937)

Bekanntlich nimmt in der Bohrschen Theorie der fastperiodischen Funktionen der Eindeutigkeitssatz oder die ihm äquivalente Parsevalsche Gleichung eine zentrale Stelle ein.

Die Verallgemeinerung von Besicowitsch bezweckt hauptsächlich, die Klasse derart zu erweitern, dass dabei die Parsevalsche Gleichung gültig bleibt.

In dieser Note will ich eine Klasse stetiger Funktionen angeben, welche die Bohrsche Klasse verallgemeinert und für welche der Eindeutigkeitssatz gilt.

Diese Funktionen genügen den folgenden zwei Bedingungen:

Bedingung I. *Für jedes* $\varepsilon > 0$ *und* $N > 0$ *kann man eine relativ dichte Zahlenmenge* $\tau = \tau(\varepsilon, N)$ *derart angeben, dass für* $|x| < N$:

$$|f(x \pm \tau) - f(x)| < \varepsilon$$

ist.

Bedingung II.

$$\tau(\varepsilon, N) + \tau(\rho, N) = \tau(\delta, N),$$

wobei $\delta = \delta(\varepsilon, \rho)$ *gegen Null strebt, falls* $\varepsilon \to 0$, $\rho \to 0$.

Offenbar genügt jede Bohrsche Funktion diesen beiden Bedingungen. Dagegen ist die Funktion

$$\frac{1}{e^{i\lambda_1 x} + e^{i\lambda_2 x} + 2},$$

wo $\frac{\lambda_1}{\lambda_2}$ irrational ist, im Bohrschen Sinne nicht fastperiodisch; aber die Bedingungen I und II sind für sie erfüllt.

Es gilt der folgende

Satz I. *Genügen* $f_1(x)$ *und* $f_2(x)$ *den Bedingungen I und II, so genügt den Bedingungen I und II auch die Funktion*

$$c_1 f_1(x) + c_2 f_2(x)$$

($c_1 c_2$ *beliebige komplexe Zahlen*) *und die Funktion*

$$f_1(x) f_2(x).$$

Wir setzen jetzt voraus, dass für die Funktion $f(x)$

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 \, dx < \infty \tag{1}$$

gilt und dass für alle reellen λ die Mittelwerte

$$a(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, l^{-i\lambda x} \, dx \tag{2}$$

existieren.

Bekanntlich kann man dann der Funktion $f(x)$ die Fouriersche Reihe

$$f(x) \sim \sum A_n \, l^{i\Lambda_n x}$$

zuordnen; A_n sind die Fourierschen Koeffizienten und Λ_n die Fourierschen Exponenten der Funktion $f(x)$.

Wir können jetzt den Fundamentalsatz formulieren.

Satz II. *Genügt $f(x)$ den Bedingungen I und II, besteht ferner die Beziehung* (1) *und existieren für alle reellen λ die Mittelwerte* (2), *so kann man für jedes $\varepsilon > 0$ und $N > 0$ ein trigonometrisches Polynom $P(x)$ derart konstruieren, dass*

$$|f(x) - P(x)| < \varepsilon$$

für $|x| < N$.

Dabei sind die Exponenten des Polynoms $P(x)$ die Fourierschen Exponenten von $f(x)$, und die Koeffizienten von $P(x)$ entstehen aus den entsprechenden Koeffizienten von $f(x)$ durch Multiplikation mit gewissen [von $f(x)$ abhängenden] Zahlen.

Aus diesem Satze und dem Satze I folgt unmittelbar der

Satz III (*Eindeutigkeitssatz*). *Genügen die Funktionen $f_1(x)$, $f_2(x)$ den Bedingungen I, II und der Beziehung* (1) *und haben sie identische Fouriersche Reihen, so sind sie identisch.*

Es sei bemerkt, dass alle hier angegebenen Resultate auf integrierbare (im Lebesgueschen Sinne) Funktionen leicht erweitert werden können.

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MATHEMATIK

ÜBER LINEARE DIFFERENTIALGLEICHUNGEN MIT FASTPERIODISCHEN KOEFFIZIENTEN

Von B. LEWITAN

(Vorgelegt von S. N. Bernstein, Mitglied d. Akademie, d. 2. X. 1937)

Es sei gegeben ein System von Differentialgleichungen

$$\frac{dx_i}{dt} = f_{i,1}(t)x_1 + f_{i,2}(t)x_2 + \ldots + f_{i,n}(t)x_n + g_i(t) \quad (i = 1, 2, \ldots, n), \text{(S)}$$

wo $f_{i,k}(t)$ und $g_i(t)$ reelle, stetige, fastperiodische Funktionen bedeuten. Zum ersten Male wurden solche Gleichungen von J. Favard (1) untersucht. Favard suchte nämlich die hinreichenden Bedingungen für den Fall, wo beschränkte Lösungen dieser Gleichungen auch fastperiodisch sind.

Die Bedingungen von J. Favard, welche wesentlich zu sein scheinen, haben den Mangel, wie dies selbst J. Favard bemerkt, dass in ihnen nicht nur die Eigenschaften des betrachteten Systems figurieren.

In dieser Note will ich zeigen, wie der Begriff der Fastperiodizität su verallgemeinern ist, um aus den Eigenschaften des Systems (S) selbst entscheiden su können, wann beschränkte Lösungen dieses Systems auch im verallgemeinerten Sinne fastperiodisch sind.

Wir nehmen eine Klasse von Funktionen, welche folgenden zwei Bedingungen genügen*:

Bedingung I. *Für jedes* $\varepsilon > 0$ *und* $N > 0$ *kann man eine relativ dichte Zahlenmenge* $\tau = \tau(\varepsilon; N)$ *derart angeben, dass für* $|x| < N$

$$|f(x \pm \tau) - f(x)| < \varepsilon$$

ist.

Bedingung II.

$$\tau(\varepsilon; N) + \tau(\rho; N) = \tau(\delta; N),$$

wobei $\delta = \delta(\varepsilon; \rho)$ *gegen Null strebt falls* $\varepsilon \to 0$, $\rho \to 0$.

Es gibt zunächst der folgende

Satz I. *Besitzt das unhomogene System*** (S) *eine beschränkte Lösung, das entsprechende homogene System hingegen keine beschränkte Lösung (die triviale Lösung* $x_i = 0$ *sei ausgeschlossen), so genügt die erwähnte beschränkte Lösung des Systems* (S) *den beiden Bedingungen I und II.*

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* Vgl. meine vorangehende Note.

** Die Koeffizienten des Systems (S) seien jetzt fastperiodisch entweder im Bohrschen oder im verallgemeinerten Sinne.

Wird ferner unter dem absoluten Betrage einer Lösung $x_i(t)$ $(i = 1, 2, \ldots, n)$ die Quadratwurzel aus

$$x_1^2(t) + x_2^2(t) + \ldots + x_n^2(t)$$

verstanden, so besteht der

Satz II. *Besitzt das homogene System keine (nichttriviale) Lösung, deren absoluter Betrag für $t \to \pm \infty$ gegen Null strebt, und besitzt das System* (S) *beschränkte Lösungen, so wird mindestens eine unter ihnen den beiden Bedingungen I und II genügen.*

Es sei bemerkt, dass es auf diesem Wege gelingt, auch neue und etwas einfachere Beweise der drei Sätze von J. Favard zu geben.

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ZITIERTE LITERATUR

[1] J. Favard, Acta Math., 51, 31—81 (1928).

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MATHÉMATIQUES

SUR UN CAS PARTICULIER DU PROBLÈME DE RIEMANN

Par N. KOTCHIN

(Présenté par I. M. Vinogradow, de l'Académie, le 4. X. 1937)

Désignons par W_1 et W_2 deux matrices arbitraires de degré n, dont les éléments sont des nombres complexes. Soit encore $Y(x)$ une matrice du même degré n, fonction de la variable complexe x.

Désignons par $\Psi_b\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right)$ une matrice $Y(x)$, définie par les propriétés suivantes:

1) $Y(x)$ est une fonction analytique de x, dont les singularités se trouvent aux points a_1, a_2, ∞;

2) en traçant sur le plan de la variable x deux coupures (a_1, ∞), (a_2, ∞), qui n'ont aucun point d'intersection et qui ne passent pas par le point b, on obtient une branche uniforme de la fonction $Y(x)$; cette branche se réduit à la matrice identique I au point $x = b$, et dans le voisinage des points singuliers a_1, a_2, ∞ on a les représentations suivantes:

$$Y(x) = (x - a_1)^{W_1} Y_1(x),$$
$$Y(x) = (x - a_2)^{W_2} Y_2(x),$$
$$Y(x) = \left(\frac{1}{x}\right)^{W_\infty} Y_\infty(x),$$

les matrices $Y_1(x)$, $Y_2(x)$, $Y_\infty(x)$, ainsi que les matrices inverses $Y_1(x)^{-1}$, $Y_2(x)^{-1}$, $Y_\infty(x)^{-1}$, étant des fonctions holomorphes de x dans le voisinage des points singuliers a_1 resp. a_2 et ∞.

Au lieu de supposer $Y(b) = I$ on peut exiger que $Y(x)$ possède dans le voisinage du point singulier a_1 une représentation de la forme:

$$Y(x) = (x - a_1)^{W_1} Y_1(x),$$

où les matrices $Y_1(x)$ et $Y_1(x)^{-1}$ sont des fonctions holomorphes de x dans le point $x = a_1$ et se réduisent à la matrice identique I dans ce point. Dans ce cas nous désignons la matrice $Y(x)$ par $\vartheta_1\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right)$. La matrice $\vartheta_2\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right)$ a une définition analogue.

Introduisons maintenant la notation suivante:

$$[W_1 W_2] = W_1 W_2 - W_2 W_1.$$

Le but de cette note est de trouver les fonctions $\Psi_b\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right)$ et $\vartheta_j'\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right)$ dans le cas particulier, où les matrices W_1 et W_2 sont assujetties aux conditions suivantes:

$$\left[W_1[W_1W_2]\right] = 0, \qquad \left[W_2[W_1W_2]\right] = 0. \tag{1}$$

Ce problème est évidemment un cas très particulier du problème de Riemann.

Remarquons en premier lieu, en supposant toujours que les matrices W_1 et W_2 satisfont aux conditions (1), que l'on a la relation suivante:

$$f(W_1)W_2 = W_2 f(W_1) + [W_1W_2]f'(W_1),$$

où $f(W_1)$ est une fonction analytique de la matrice W_1; notons encore les formules:

$$e^{W_1}e^{W_2} = e^{W_2}e^{W_1}e^{[W_1W_2]},$$

$$e^{W_1}e^{W_2} = e^{W_1+W_2+\frac{1}{2}[W_1W_2]};$$

pour la démonstration de la dernière formule on peut, par exemple, considérer les deux fonctions:

$$\Phi(\alpha) = e^{\alpha W_1}e^{W_2}, \quad \Psi(\alpha) = e^{\alpha W_1+W_2+\frac{1}{2}\alpha[W_1W_2]}.$$

Ces fonctions prennent la même valeur e^{W_2} pour $\alpha = 0$ et satisfont à la même équation différentielle:

$$\frac{d\Phi}{d\alpha} = W_1\Phi, \quad \frac{d\Psi}{d\alpha} W_1\Psi,$$

par suite, ces fonctions sont identiques.

Notons encore la formule suivante

$$\frac{d}{dx} e^{W_1a(x)+W_2b(x)+[W_1W_2]c(x)} =$$

$$= e^{W_1a(x)+W_2b(x)+[W_1W_2]c(x)} \left\{W_1\frac{da}{dx}+W_2\frac{db}{dx}+[W_1W_2]\left(\frac{dc}{dx}+\frac{1}{2}b\frac{da}{dx}-\frac{1}{2}a\frac{db}{dx}\right)\right\}.$$

Introduisons enfin la notation:

$$L_b(x, a) = \int_b^x \frac{dx}{x-a} = \lg\frac{x-a}{b-a}.$$

Il est maintenant facile de vérifier que la fonction

$$\Psi_b\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right) =$$

$$= e^{W_1L_b(x,a_1)+W_2L_b(x,a_2)+[W_1W_2]\left\{\frac{1}{2}L_b(x,a_1)L_b(x,a_2)-L_b(a_2,a_1)L_b(x,a_2)-\int_b^x \frac{L_{a_1}(x,a_2)}{x-a_1}dx\right\}}$$

satisfait à toutes les conditions imposées.

Le calcul montre encore que pour la branche de la fonction $\Psi_b\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix} \middle| x\right)$, qui s'obtient par le prolongement analytique dans le domaine qui se trouve à gauche de la coupure (a_1, ∞), on a:

$$W_\infty = -W_1 - W_2 + \pi i[W_1W_2];$$

de même quand on prolonge la fonction $\Psi_b\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix}\middle| x\right)$ dans le domaine qui se trouve à droite de la coupure (a_1, ∞), on a:

$$W_\infty = -W_1 - W_2 - \pi i\,[W_1 W_2].$$

Pour la matrice $\vartheta_1\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix}\middle| x\right)$ qui est égale au produit $\Psi_b\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix}\middle| x\right)C$, où C est une matrice constante, on trouve la représentation:

$$\vartheta_1\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix}\middle| x\right) =$$

$$= (x-a_1)^{W_1} e^{W_2 L_{a_1}(x,\,a_2) - [W_1 W_2]\left\{\lg(a_2-a_1)\,L_{a_1}(x,\,a_2) + \int\limits_{a_1}^{x} \frac{L_{a_1}(x,\,a_2)}{x-a_1}\,dx\right\}}$$

De même on a:

$$\vartheta_2\left(\begin{matrix} W_1 & W_2 \\ a_1 & a_2 \end{matrix}\middle| x\right) =$$

$$= (x-a_2)^{W_2} e^{W_1 L_{a_2}(x,\,a_1) + [W_1 W_2]\left\{\lg(a_1-a_2)\,L_{a_2}(x,\,a_1) + \int\limits_{a_2}^{x} \frac{L_{a_2}(x,\,a_1)}{x-a_2}\,dx\right\}}$$

La matrice $\Psi_b(x)$ satisfait à l'équation différentielle de Gauss:

$$\frac{d\Psi_b}{dx} = \Psi_b\left(\frac{U_1}{x-a_1} + \frac{U_2}{x-a_2}\right),$$

où

$$U_1 = W_1 + [W_1 W_2]\,L_b(a_1,\ a_2),$$
$$U_2 = W_2 - [W_1 W_2]\,L_b(a_2,\ a_1).$$

Les matrices $\vartheta_1(x)$ et $\vartheta_2(x)$ satisfont à la même équation; pour la matrice $\vartheta_1(x)$ les matrices U_1 et U_2 ont les valeurs:

$$U_1 = W_1,\ U_2 = W_2 - [W_1 W_2]\lg(a_2 - a_1),$$

et pour la matrice $\vartheta_2(x)$ ces valeurs sont

$$U_1 = W_1 + [W_1 W_2]\lg(a_1 - a_2),\quad U_2 = W_2.$$

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MATHÉMATIQUES

SUR LA REPRÉSENTATION GÉNÉRALE DES SOLUTIONS DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES DU SECOND ORDRE

Par I. VÈCOUA

(Présenté par I. M. Vinogradow, de l'Académie, le 31. VIII. 1937)

D'après les données littéraires on obtient au moyen de deux fonctions analytiques d'une variable complexe une représentation intégrale de la solution générale de l'équation

$$L(U) = U_{z\bar{z}} = AU_z + BU_{\bar{z}} + CU = 0, \tag{1}$$

où

$$z = z_1 + iz_2, \quad \bar{z} = \bar{z}_1 - i\bar{z}_2, \quad U_z = \frac{1}{2}\left(\frac{\partial U}{\partial z_1} - i\frac{\partial U}{\partial z_2}\right),$$
$$U_{\bar{z}} = \frac{1}{2}\left(\frac{\partial U}{\partial z_1} + i\frac{\partial U}{\partial z_2}\right), \quad z_k = x_k + iy_k \quad (k = 1, 2),$$

A, B et C étant des fonctions données définies dans un domaine (à quatre dimensions) de l'espace des variables z_1 et z_2. D'ailleurs, dans la représentation indiquée se trouve un noyau qui est une solution particulière d'une certaine équation différentielle du second ordre à trois variables indépendantes.

Dans cette note nous indiquons une autre méthode pour représenter la solution générale de l'équation (1) au moyen d'un certain opérateur linéaire intégral de seconde espèce dépendant de deux fonctions analytiques d'une variable complexe. Notre méthode n'exige pas la construction d'une solution particulière de l'équation précédemment indiquée et le problème se résoud complètement au moyen d'une équation intégrale du type de Volterra dans un domaine complexe à deux dimensions.

Dans la suite nous allons supposer que les fonctions A, B et C sont analytiques et regulières dans le voisinage de l'origine des coordonnées dans l'espace à quatre dimensions des variables z_1 et z_2.

En intégrant les deux membres de l'équation (1) deux fois, par rapport à z d'abord et à $\bar{z}$ ensuite, et en intégrant par parties, on obtient une équation intégrale de Volterra qui équivaut à l'équation différentielle (1) initiale:

$$U(z, \bar{z}) + \int_0^{\bar{z}} A(z, \bar{\xi})\, U(z, \bar{\xi})\, d\bar{\xi} + \int_0^{z} B(\xi, \bar{z})\, U(\xi, \bar{z})\, d\xi +$$
$$+ \int_0^{z} d\xi \int_0^{\bar{z}} D(\xi, \bar{\xi})\, U(\xi, \bar{\xi})\, d\bar{\xi} = \varphi(z) + \bar{\psi}(\bar{z}), \tag{2}$$

où

$$D(z,\bar{z}) = C(z,\bar{z}) - \frac{\partial A(z,\bar{z})}{\partial z} - \frac{\partial B(z,z)}{\partial \bar{z}}, \tag{3}$$

$\varphi(z)$ et $\bar{\psi}(\bar{z})$ étant des fonctions analytiques des variables z et $\bar{z}$ respectivement; d'ailleurs il est facile à voir que l'on a:

$$\varphi'(z) = U_z(z,0) + B(z,0)\,U(z,0)$$

et

$$\bar{\psi}'(\bar{z}) = U_{\bar{z}}(0,\bar{z}) + A(0,\bar{z})\,U(0,\bar{z}).$$

L'équation intégrale (2) est ce que l'on appelle une équation de Volterra chargée. Mais, une tranformation simple permet de la réduire à la forme ordinaire.

Soit, en effèt,

$$U(z,\bar{z}) + \int_0^{\bar{z}} A(z,\bar{\xi})\,U(z,\bar{\xi})\,d\bar{\xi} = U_0(z,\bar{z}). \tag{4}$$

En différentiant par rapport à $\bar{z}$, on a:

$$\frac{\partial U(z,\bar{z})}{\partial \bar{z}} + A(z,\bar{z})\,U(z,\bar{z}) - \frac{\partial U_0(z,\bar{z})}{\partial \bar{z}} = 0. \tag{5}$$

Il est facile à voir que l'intégrale générale de cette équation est

$$U(z,\bar{z}) = \exp\left[-\int_0^{\bar{z}} A(z,\bar{\xi})\,d\bar{\xi}\right]\left\{U(z,0) + \right.$$

$$\left. + \int_0^{\bar{z}} \frac{\partial U_0(z,\bar{\xi})}{\partial \bar{\xi}} \exp\left[\int_0^{\bar{z}} A(z,\bar{\xi})\,d\bar{\xi}_1\right] d\bar{\xi}\right\}$$

ou, en tenant compte de ce que, en vertu de (4),

$$U(z,0) = U_0(z,0)$$

et en intégrant par parties,

$$U(z,\bar{z}) = U_0(z,\bar{z}) - \int_0^{\bar{z}} A(z,\bar{\xi})\,U_0(z,\bar{\xi}) \exp\left[-\int_{\bar{\xi}}^{\bar{z}} A(z,\bar{\xi}_1)\,d\bar{\xi}_1\right] d\bar{\xi}. \tag{6}$$

En substituant dans l'équation (2) l'expression de $U(z,\bar{z})$, tirée de (6), on obtient:

$$U_0(z,\bar{z}) + \int_0^{z} B(\xi,\bar{z})\,U_0(\xi,\bar{z})\,d\xi +$$

$$+ \int_0^{z} d\xi \int_0^{\bar{z}} D_1(\bar{z},\xi,\bar{\xi})\,U_0(\xi,\bar{\xi})\,d\bar{\xi} = \varphi(z) + \bar{\psi}(\bar{z}), \tag{7}$$

où

$$D_1(\bar{z},\xi,\bar{\xi}) = -B(\xi,\bar{z})\,A(\xi,\bar{\xi}) \exp\left[-\int_{\bar{\xi}}^{\bar{z}} A(\xi,\bar{\xi}_1)\,d\bar{\xi}_1\right] + D(\xi,\bar{\xi}) -$$

$$- A(\xi,\bar{\xi}) \int_{\bar{\xi}}^{\bar{z}} D(\xi,\bar{\xi}_1) \exp\left[-\int_{\bar{\xi}}^{\bar{\xi}_1} A(\xi_1,\bar{\xi}_2)\,d\bar{\xi}_2\right] d\bar{\xi}_1. \tag{8}$$

Posons maintenant:

$$U_0(z, \bar{z}) + \int_0^z B(\xi, \bar{z})\, U_0(\xi, \bar{z})\, d\xi = V(z, \bar{z}). \tag{9}$$

On obtient alors

$$U_0(z, \bar{z}) = V(z, \bar{z}) - \int_0^z B(\xi, \bar{z})\, V(\xi, \bar{z}) \exp\left[-\int_\xi^z B(\xi_1, \bar{z})\, d\xi_1\right] d\xi. \tag{10}$$

En vertu de (9) et (10) l'équation (7) prend la forme:

$$V(z, \bar{z}) = \int_0^z d\xi \int_0^{\bar{z}} K(z, \bar{z}, \xi, \bar{\xi}) V(\xi, \bar{\xi})\, d\bar{\xi} + \varphi(z) + \bar{\psi}(\bar{z}), \tag{11}$$

où

$$K(z, \bar{z}, \xi, \bar{\xi}) = -D_1(\bar{z}, \xi, \bar{\xi}) +$$

$$+ B(\xi, \bar{\xi}) \int_\xi^z D_1(\bar{z}, \xi_1, \bar{\xi}) \exp\left[-\int_\xi^z B(\xi_2, \bar{\xi})\, d\xi_2\right] d\xi_1. \tag{12}$$

C'est une équation de Volterra ordinaire.

Soit $H(z, \bar{z}, \xi, \bar{\xi})$ la résolvante du noyau $K(z, \bar{z}, \xi, \bar{\xi})$.

On a

$$H(z, \bar{z}, \xi, \bar{\xi}) = \sum_{n=1}^{\infty} K_n(z, \bar{z}, \xi, \bar{\xi}), \tag{13}$$

où

$$K_1 = K(z, \bar{z}, \xi, \bar{\xi}),\ K_n = \int_\xi^z d\xi_1 \int_{\bar{\xi}}^{\bar{z}} K(z, \bar{z}, \xi_1, \bar{\xi}_1)\, K_{n-1}(\xi_1, \bar{\xi}_1, \xi, \bar{\xi})\, d\bar{\xi}_1. \tag{14}$$

$(n=1,\ 2,\ 3\ldots)$

Alors la solution de l'équation intégrale (11) est

$$V(z, \bar{z}) = \varphi(z) + \bar{\psi}(\bar{z}) + \int_0^z H_1(z, \bar{z}, \xi)\, \varphi(\xi)\, d\xi + \int_0^{\bar{z}} H_2(z, \bar{z}, \bar{\xi})\, \bar{\psi}(\bar{\xi})\, d\bar{\xi}, \tag{15}$$

où

$$H_1(z, \bar{z}, \xi) = \int_0^{\bar{z}} H(z, \bar{z}, \xi, \bar{\xi})\, d\bar{\xi};\quad H_2(z, \bar{z}, \bar{\xi}) = \int_0^z H(z, \bar{z}, \xi, \bar{\xi})\, d\xi. \tag{16}$$

Exprimons maintenant la solution de l'équation intégrale initiale (2) au moyen des fonctions $\varphi(z)$ et $\bar{\psi}(\bar{z})$.

En vertu de (15) on obtient de la formule (10) après avoir fait des transformations simples:

$$U_0(z, \bar{z}) = \varphi(z) + \bar{\psi}(\bar{z}) \exp\left[-\int_0^z B(\xi, \bar{z})\, d\xi\right] +$$

$$+ \int_0^{\bar{z}} H_3(z, \bar{z}, \xi)\, \varphi(\xi)\, d\xi + \int_0^{\bar{z}} H_4(z, \bar{z}, \bar{\xi})\, \bar{\psi}(\bar{\xi})\, d\bar{\xi}, \tag{17}$$

où

$$H_3(z, \bar{z}, \xi) = H_1(z, \bar{z}, \xi) - B(\xi, \bar{z}) \exp\left[-\int_{\xi}^{z} B(\xi_1, \bar{z})\, d\xi_1\right] -$$

$$- \int_{\xi}^{z} B(\xi_1, \bar{z})\, H_1(\xi_1, \bar{z}, \xi) \exp\left[-\int_{\xi}^{z} B(\xi_2, \bar{z})\, d\xi_2\right] d\xi_1,$$

$$H_4(z, \bar{z}, \bar{\xi}) = H_2(z, \bar{z}, \bar{\xi}) - \int_{0}^{z} B(\xi, \bar{z})\, H_2(\xi, \bar{z}, \bar{\xi}) \exp\left[-\int_{\xi}^{z} B(\xi_1, \bar{z})\, d\xi_1\right] d\xi.$$

En substituant la valeur de $U_0(z, \bar{z})$ dans la formule (6) et en effectuant quelques calculs simples on obtient la formule cherchée:

$$U(z, \bar{z}) = \varphi(z) \exp\left[-\int_{0}^{\bar{z}} A(z, \bar{\xi})\, d\bar{\xi}\right] + \bar{\psi}(\bar{z}) \exp\left[-\int_{0}^{z} B(\xi, \bar{z})\right] d\xi +$$

$$+ \int_{0}^{z} H^{(2)}(z, \bar{z}, \bar{\xi})\, \varphi(\xi)\, d\xi + \int_{0}^{\bar{z}} H^{(2)}(z, \bar{z}, \bar{\xi})\, \bar{\psi}(\bar{\xi})\, d\bar{\xi}, \qquad (18)$$

où

$$H^{(2)}(z, \bar{z}, \bar{\xi}) = H_3(z, \bar{z}, \xi) - \int_{0}^{\bar{z}} A(z, \bar{\xi})\, H_3(z, \bar{\xi}, \xi) \exp\left[-\int_{\bar{\xi}}^{\bar{z}} A(z, \bar{\xi}_1)\, d\bar{\xi}_1\right] d\bar{\xi},$$

$$H^{(2)}(z, \bar{z}, \bar{\xi}) = H_4(z, \bar{z}, \bar{\xi}) - A(z, \bar{\xi}) \exp\left[-\int_{0}^{z} B(\xi, \bar{\xi})\, d\xi - \int_{\bar{\xi}}^{\bar{z}} A(z, \bar{\xi})\right] d\bar{\xi} -$$

$$- \int_{\bar{\xi}}^{\bar{z}} A(z, \bar{\xi})\, H_4(z, \bar{\xi}_1, \xi) \exp\left[-\int_{\bar{\xi}_1}^{\bar{z}} A(z, \bar{\xi}_2)\, d\bar{\xi}_2\right] d\bar{\xi}_1.$$

La formule (18) donne la solution de l'équation intégrale (2) et en même temps la solution générale de l'équation différentielle (1) qui lui est équivalente. Le point essentiel du procédé précédemment indiqué pour résoudre l'équation (1) est la résolution de l'équation intégrale (11) de Volterra, mais il est toujours possible de la résoudre, par exemple, au moyen d'approximations successives.

Comme exemple, on peut considérer le cas où les coefficients A, B et C de l'équation (1) sont des constantes. Il est aisé de voir que dans ce cas le noyau de l'équation intégrale (11) est

$$K(z, \bar{z}, \xi, \bar{\xi}) = (C - AB) \exp\left[-A(\bar{z} - \bar{\xi}) - B(z - \xi)\right], \qquad (19)$$

et la résolvante $H(z, \bar{z}, \xi, \bar{\xi})$ est, en vertu de (13) et (14), égale à

$$H(z, \bar{z}, \xi, \bar{\xi}) = -(C - AB) \exp\left[-A(\bar{z} - \bar{\xi}) - B(z - \xi)\right] J_0\left(2\sqrt{(C - AB)(z - \xi)(\bar{z} - \bar{\xi})}\right). \qquad (20)$$

On obtient la solution de l'équation différentielle (1) au moyen de la formule (18).

En particulier, si $C = A \cdot B$, on a $H(z, \bar{z}, \xi, \bar{\xi}) = 0$ et, comme il est aisé de voir, en vertu des formules (17) et (18), la solution de l'équation différentielle (1) prend dans ce cas la forme:

$$U(z, \bar{z}) = e^{-A\bar{z}} \varphi(z) + e^{-Bz} \bar{\psi}(\bar{z}) - Be^{-A\bar{z}} \int_0^z e^{-B(z-\xi)} \varphi(\xi)\, d\xi -$$

$$- Ae^{-Bz} \int_0^{\bar{z}} e^{-A(\bar{z}-\bar{\xi})} \bar{\psi}(\bar{\xi})\, d\bar{\xi}. \qquad (21)$$

Indiquons enfin encore une application de notre méthode à un problème que nous appellerons problème de Goursat. Introduisons à cet effet les notions préliminaires suivantes. Nous donnerons aux multiplicités dans l'espace à quatre dimensions $\{z, \bar{z} = \text{const}\}$ et $\{z = \text{const}, \bar{z}\}$ le nom des caractéristiques de l'équation différentielle (1).

Considérons dans chaque multiplicité une caractéristique et prenons une valeur de la fonction $U(z, \bar{z})$ sur chacune d'elles. Le problème de Goursat consiste en ce qu'il faut déterminer à partir de ces valeurs données la valeur de $U(z, \bar{z})$ en tous les points qui ne se trouvent pas sur ces caractéristiques mais sont assez proches à leur point d'intersection. Ce problème peut être énoncé brièvement ainsi:

Trouver la solution de l'équation différentielle (1) vérifiant les conditions

$$U(z, 0) = f(z), \quad U(0, \bar{z}) = \bar{g}(\bar{z}), \quad (\bar{g}(0) = f(0)), \qquad (22)$$

où $f(z)$ et $\bar{g}(\bar{z})$ sont des fonctions analytiques et régulières des variables z et $\bar{z}$ respectivement au voisinage de l'origine des coordonnées.

La solution de ce problème se trouve dans la formule (18), si l'on remplace dans cette formule $\varphi(z)$ et $\bar{\psi}(\bar{z})$ par les expressions:

$$\varphi(z) = f(z) + \int_0^z B(z, 0) f(z)\, dz,$$

$$\bar{\psi}(\bar{z}) = \bar{g}(\bar{z}) + \int_0^{\bar{z}} A(0, \bar{z})\, \bar{g}(\bar{z})\, d\bar{z}.$$

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MÉCANIQUE CÉLESTE

SUR LE PROBLÈME DE LA LOCALISATION DES TRAJECTOIRES DYNAMIQUES DANS L'ESPACE DES PHASES

Par N. MOISSEIEV

(Présenté par V. G. Fessenkov, de l'Académie, le 14. X. 1937)

Le problème de l'étude quantitative des mouvements définis par un système des équations différentielles:

$$\frac{dx_k}{dt} = X_k(x_1, x_2, \ldots, x_n) \qquad (k = 1, 2, \ldots, n), \tag{1}$$

peut être considéré à deux points de vue. On peut chercher des expressions analytiques exactes des coordonnées des phases x_k en fonctions du temps t, vérifiant les équations (1). C'est un problème d'intégration du système (1). Mais on peut ausssi chercher des domaines ou, si l'on veut, des «tubes» dans l'espace des phases $(x_1, x_2, \ldots, x_n)$ qui déterminent la trajectoire du système à la moitié de la largeur du «tube» près. Nous nommons ce dernier problème—le problème de la localisation des trajectoires.

Ce deuxième aspect du problème de l'étude quantitative des trajectoires dynamiques présente un intérêt considérable pratique ainsi que méthodique. Il est à noter que c'est sous cet aspect qu'on peut compter sur une synthèse féconde réunissant les méthodes qualitatives et les théories quantitatives des problèmes dynamiques (1).

La direction particulière dans la théorie de la localisation qui nous intéresse dans la note présente est bien caractérisée par les indications des théorèmes de Perron (2), Caplygin (3) et de Cetaïew (4). Les considérations élémentaires de cette note ne sont que des compléments tout à fait naturels aux trois théorèmes cités.

Soit S une certaine surface qui est fermée, continue et qui limite dans l'espace des phases $(x_1, x_2, \ldots, x_n)$ un volume T à n dimensions simplement connexe. Nous supposons cette surface S d'être composée de trois morceaux suivants:

S_1, un morceau simplement connexe qui fait une partie de la surface

$$U_1(x_1, x_2, \ldots, x_n) = 0, \tag{2}$$

S_2, un morceau simplement connexe, n'ayant pas de points communs avec S_1 et faisant partie de la surface

$$U_2(x_1, x_2, \ldots, x_n) = 0, \tag{3}$$

et enfin S_3, un morceau dont la somme avec les morceaux S_1 et S_2 donne la surface totale S et lequel nous considérons comme une partie de la surface

$$W(x_1, x_2, \ldots, x_n) = 0. \tag{4}$$

Nous supposons que les fonctions U_1, U_2 et W, entrant dans les équations (2), (3) et (4), sont des fonctions continues et uniformes des variables $x_1, x_2, \ldots, x_n$, admettant sur S_1, S_2 et S_3 respectivement des dérivées partielles du premier ordre par rapport à ces variables. Les dérivées nommées sont supposées d'être bornées par les valeurs absolues; elles peuvent avoir des discontinuités finies aux points de certaines courbes analytiques tracées en nombre fini sur la surface S. Les équations des surfaces auxiliaires (2), (3) et (4) sont supposées écrites d'une telle manière que les cosinus directeurs de la normale extérieure par rapport au volume T ont toujours le même signe que les dérivées partielles correspondantes des fonctions U_1, U_2 et W. La courbe frontière séparant S_1 de S_3 est comprise dans l'ensemble S_1 ainsi que dans l'ensemble S_3. La remarque analogue est valable par rapport à la frontière séparant S_2 de S_3. Nous nommerons l'ensemble S_3 surface latérale du tube des phases T; les ensembles S_1 et S_2, première et deuxième sections du même tube T.

Dans le cas particulier, quand la première section du tube T se dégénère en un seul point A, nous parlerons d'un tube conique des phases, ayant son sommet au point A. Dans le cas où la première section coïncide avec la deuxième, d'une manière que les normales extérieures aux surfaces S_1 et S_2 aux points coïncidants deviennent opposées l'une à l'autre, nous parlerons d'un tube fermé. Dans ce cas notre volume T cesse d'être un volume simplement connexe et devient un volume doublement connexe, equivalent, au point de vue de la topologie, au torus.

Tous les tubes mentionnés dans cette note (sauf un seul cas, pour lequel nous faisons des indications correspondantes) sont supposés de ne pas avoir, ni dans leur intérieur, ni sur leur surface frontière, des points singuliers du système (1), dans lesquels cessent d'être remplies les conditions de la détermination univoque de la trajectoire par son point initial. Les fonctions X_k sont supposées continues et uniformes dans les domaines considérés.

Formons les expressions:

$$(\dot{U}_1)_{S_1} = \Big(\sum_{k=1}^{n} \frac{\partial U_1}{\partial x_k} X_k\Big)_{S_1} \tag{5}$$

et

$$(\dot{W})_{S_3} = \Big(\sum_{k=1}^{n} \frac{\partial W}{\partial x_k} X_k\Big)_{S_3}. \tag{6}$$

En vertu du susdit, se sont deux fonctions uniformes des variables x_1, x_2, $\ldots$ x_n, définies sur les ensembles S_1 et S_3 resp. et continues partout, sauf peut-être les points des ensembles de mesure nulle par rapport à la mesure de S_1 ou de S_3, dans lesquels les fonctions (5) et (6) peuvent avoir des discontinuités finies.

Si ces fonctions $(\dot{U}_1)_{S_1}$ et $(\dot{W})_{S_3}$ sont des fonctions partout négatives (positives) et ne s'annulant pas dans aucun point des ensembles S_1 et S_3 resp., nous dirons que le tube T est borné dedans (dehors) par les morceaux S_1 et S_3 de sa surface frontière S.

Théorème 1. *Si le système* (1) *admet un invariant intégral*

$$\int\int \ldots \int M(x_1, x_2, \ldots x_n, t)\, dx_1 dx_2 \ldots dx_n, \tag{7}$$

où M reste positive et bornée pour toutes les valeurs considérées de x_1, x_2, ..., x_n et pour t quelconque, et si le tube du type cylindrique T est borné dedans (dehors) par sa surface latérale S_3 et par sa section première S_1, il existe un nombre infini des segments des trajectoires, qui n'ont pas de points communs avec la surface latérale S_3 et qui joignent les points de la première section S_1 avec ceux de la deuxième section S_2, étant compris dans l'intérieur du tube T. La direction du mouvement le long de ces segments des trajectoires est la direction de S_1 à S_2 (de S_2 à S_1). La mesure de l'ensemble de points de S_1, correspondant aux trajectoires exceptionnelles, est nulle par rapport à la mesure de l'ensemble de points de S_1, qui déterminent les trajectoires dont l'existence est assurée par le théorème.

Sans nous occuper de la démonstration évidente de ce théorème, nous considérerons une autre notion auxiliaire qui nous fournira un autre théorème, semblable au théorème 1. Soit

$$V(x_1, x_2, \ldots, x_n) \tag{8}$$

une certaine fonction des variables $(x_1, x_2, \ldots, x_n)$, qui est uniforme et continue partout dans l'intérieur et aux frontières du tube T et qui admet dans le même domaine des dérivées partielles du premier ordre par rapport aux $x_1, x_2, \ldots, x_n$. Ces dérivées sont supposées continues partout à l'intérieur du domaine envisagé, mais elles peuvent aussi avoir des discontinuités finies dans les points d'un ensemble de mesure nulle par rapport à la mesure des l'ensemble total, sur lequel nous définissons notre fonction (8). Le système des morceaux des surfaces de la famille uniparamétrique

$$V(x_1, x_2, \ldots, x_n) = c, \tag{9}$$

situés à l'intérieur du tube T est supposé d'être équivalent (au sens topologique) au système des morceaux des plans parallèles. Nous supposons aussi que pour chaque couple de deux morceaux distincts des surfaces (9)

$$V = c_1 \tag{10}$$

et

$$V = c_2 \tag{11}$$

dont (10) sépare (11) de S_1 tandis que (11) sépare (10) de S_2, les valeurs du paramètre c vérifient toujours l'inégalité

$$c_1 < c_2. \tag{12}$$

Nous supposons enfin que les sections S_1 et S_2 du tube T sont des éléments de l'ensemble de morceaux des surfaces (9).

Dans le cas où toutes les conditions mentionnée sont remplies, nous nommons (9) un système de sections régulières du tube T, orienté dans la direction de S_1 à S_2.

Théorème 2. *Si le système* (1) *permet la construction d'un tel système* (9) *de sections régulières du tube T, orienté dans la direction de S_1 à S_2, que l'expression*

$$\dot{V} = \sum_{k=1}^{n} \frac{\partial V}{dx_k} X_k \tag{13}$$

soit distincte de zéro et positive partout à l'intérieur et à la frontière du tube T et si le tube T est borné dedans par les morceaux S_1 et S_3 de sa surface frontière S, chaque trajectoire partant d'un point quelconque de S_1 contient un segment, joignant le point nommé avec un des points de S_2 et n'ayant pas de points communs avec la surface latérale S_3, étant situé entièrement dans l'intérieur du tube T.

Ce théorème, formulé pour le cas du tube «cylindrique» est valide aussi dans le cas du tube de type conique, dont le sommet est la section S_1, dégénérée en un point. Dans le cas où ce sommet est un point singulier du système (1), dans lequel toutes les fonctions X_k s'annulent, le théorème 2 coïncide, mutatis mutandis, avec un cas particulier du théorème de Četaïew [4]. Étant donné cela, le théorème 2 doit être considéré comme une variante naturelle du théorème de cet éminent géomètre de Kazan. Le théorème de Četaïew étant employé avec beaucoup de succès dans la théorie de la stabilité au sens de Liapounoff, notre variante a trouvé son application utile dans la théorie de la stabilité du troisième type (stabilité dans le cas des perturbations finies, ou la stabilité au sens de Kryloff et Bogoliouboff).

Théorème 3. *Si le tube T vérifiant toutes les conditions du théorème 2 est en même temps le tube fermé, il existe une (au moins) trajectoire fermée du système* (1), *située avec tous ses points à l'intérieur du tube T et faisant une simple révolution (c'est-à-dire, ayant avec chaque morceau des surfaces* (9) *un seul point commun).*

Cette trajectoire périodique doit être asymptotiquement stable au sens de A. Liapounoff.

Si le tube T est borné par S_1 et S_3 dehors, les théorèmes 2 et 3 sont aussi valides. La trajectoire périodique du théorème 3 sera dans ce cas absolument instable au sens de Liapounoff.

Les théorèmes énoncés ci-dessus sont sans doute d'une nature bien élémentaire et presque triviale. Les conditions de ces théorèmes, rédigées par nous d'une manière à rendre tout à fait évidente leur suffisance, peuvent être modifiées. Cela peut nous fournir de nouveaux théorèmes, renfermant des conditions moins compliquées et suffisantes pour l'existence au moins d'une trajectoire comprise dans le tube considéré.

Remarquons enfin, qu'il est possible de rédiger aussi des théorèmes qui sont dans un certain sens réciproques par rapport aux théorèmes susdits.

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ASTROPHYSICS

ON THE ORIGIN OF STELLAR ENERGY

By L. LANDAU

(Communicated by L. I. Mandelstam, Member of the Academy, 15. XI. 1937)

It is well known that matter consists of nuclei and electrons. Nevertheless it can be shown that in bodies of very large mass this usual „electronic" state of matter can become unstable. The reason for this lies in the fact that the „electronic" state of matter does not lead to extremely great densities, because at such densities electrons form a Fermi gas having an immense pressure. On the other hand, it is easy to see that matter can go into another state which is much more compressible—the state where all the nuclei and electrons have combined to form neutrons. Even if we assume that neutrons repel each other, this repulsion can become appreciable only at densities of the order of magnitude of nuclear densities, i.e. 10^{14} g/cm^3, and the pressure of a Fermi gas consisting of neutrons is much less than an electronic gas of the same density, because of the greater mass of the neutrons.

Therefore in spite of the fact that the „neutronic" state of matter is, in usual conditions, energetically less favourable, since the reaction of neutron formation is strongly endothermic, this state must nevertheless become stable when the mass of the body is large enough. In this case the gravitational energy gained in going over to the neutronic state with its greater density, compensates the losses of internal energy.

It is easy to compute the critical mass of the body for which the „neutronic" state begins to be more stable than the „electronic" state. First of all we must calculate the energy necessary to form one neutron. For instance in the reaction ${}^{8}_{16}O + 8\bar{l} = 16\,{}^{0}_{1}n$ we find from the mass defects that to form one neutron the energy required is 0.008 mass units or $1 \cdot 2 \cdot 10^{-5}$ ergs (7.5 MV). To transfom one gram of matter into neutrons we thus need $7 \cdot 10^{18}$ erg/g.

Now we must calculate the gain in gravitational energy. The gravitational energy of the much less dense «electronic» state can, of course, be neglected. Let us assume first of all that the neutronic state has a constant density 10^{14} g/cm^3. The gravitational energy of a homogeneous sphere of mass M is then

$$3 \cdot 10^{-3} M^{5/3} \text{ ergs.}$$

For the stability of the neutronic phase we must then have

$$3 \cdot 10^{-3} M^{5/3} > 7 \cdot 10^{18} M,$$

or

$$M > 10^{32}\text{g} = 0.05\odot.$$

On the other hand, if we assume that the neutrons behave like a Fermi gas we find for the energy

$$4 \cdot 10^{-22} M^{7/3} \text{ ergs}$$

and, hence,

$$M > 1.5 \cdot 10^{30}\text{g} = 10^{-3}\odot$$

which critical value is even less than on the first assumption.

When the mass of the body is greater than the critical mass, then in the formation of the „neutronic" phase an enormous amount of energy is liberated, and we see that the conception of a „neutronic" state of matter gives an immediate answer to the question of the sources of stellar energy. The sun during its probable time of radiation (about $2 \cdot 10^9$ years according to general relativity theory) must have emitted something of the order of magnitude of $3 \cdot 10^{50}$ ergs. The liberation of this amount of energy requires the transition of only about 2% of the mass of the sun (with the assumption of constant density) or even only $3 \cdot 10^{-3}\odot$ (with the Fermi gas model) into the „neutronic" phase. Even for such a bright star as β-Orioni we find for the mass of the neutronic core only about $0.1\odot$ (with the Fermi gas model).

Thus we can regard a star as a body which has a neutronic core whose steady growth liberates the energy which maintains the star at its high temperature; the condition at the boundary between the two phases is as usual the equality of chemical potentials. The detailed investigation of such a model should make possible the construction of a consistent theory of stars.

As regards the question of how the initial core is formed, the author has shown in a previous article* that the formation of a core must certainly take place in a body with a mass greater than $1.5\odot$. In stars with smaller mass the conditions which could make possible the formation of the initial core have yet to be made clear.

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PHYSIK

BEITRÄGE ZUR THEORIE DER BROWNSCHEN BEWEGUNG

Von N. WASSMUTH

(Vorgelegt von S. I. Wawilov, Mitglied der Akademie, d. 13. X. 1937)

Der Zweck der vorliegenden Notiz ist es für ein System von n Freiheitsgraden, das kleine Schwingungen macht, eine Anfangsverteilung (für $t = 0$) einzuführen. Zum Unterschied vom früher behandelten Falle eines «isolierten Systems» [IV](1), das jetzt zu betrachtende System wird der Wirkung «äusserer Kräfte» Q_r unterworfen. Alle «äussere Kräfte» werden als harmonische Funktionen der Zeit $Q_r = D_r e^{i\varepsilon t}$ angesetzt.

§ 1. Es werde angenommen, dass die Verteilungsfunktion für ein «äusseren Kräften» Q_r unterworfenes System im Grenzfall (für $t \to \infty$) die Maxwell-Boltzmannsche Form:

$$f(v_1, \ldots, v_n, x_1, \ldots, x_n) = \left(\frac{a}{\pi}\right)^n \omega_1 \ldots \omega_n e^{-a \sum_r^n [(v_r - v_r')^2 + \omega_r^2 (x_r - x_r')^2]}, \tag{1}$$

einnimmt, worin v_r' bzw. x_r' die Geschwindigkeits- bzw. Koordinatenwerte bei der durch die Q_r-Kräfte erzwungener Bewegung bedeuten.

Mit f von der Form (1) bilden wir die Funktion Φ [I (6)]

$$\Phi(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) = \int_{-\infty}^{+\infty} \ldots \int e^{-\sum_r^n (\lambda_r v_r + \mu_r x_r)} f \, dv_1 \ldots dv_n \, dx_1 \ldots dx_n, \tag{2}$$

es ergibt sich

$$\Phi = e^{\sum_r^n \left[\frac{\lambda_r^2}{4a} + \frac{\mu_r^2}{4a\omega_r^2} - \lambda_r v_r' - \mu_r x_r'\right]},$$

$$z = \log \Phi = \sum_{r=1}^n \left[\frac{\lambda_r^2}{4a} + \frac{\mu_r^2}{4a\omega_r^2} - \lambda_r v_r' - \mu_r x_r'\right]. \tag{3}$$

Nachdem wir für z die Form (3) wählen, erhalten wir der aus Gleichung [I (12)]:

$$\chi(\lambda_1, \ldots, \lambda_n) = \frac{1}{2a} \sum_{r,s}^n b_{rs} \lambda_r \lambda_s. \tag{4}$$

Für ein der Einwirkung «äusserer Kräfte» Q_r unterworfenes System hat die Funktion χ dieselbe Form wie für ein «isoliertes System» [IV(10)]. Folglich ist die Voraussetzung betr. der Form der Grenzverteilung (1) richtig.

§ 2. Mit dem in geschilderten Weise erhaltenen χ (4) bilden wir die Funktion [I(14)]:

$$z = \frac{1}{2i} \sum_{r,s}^{n} b_{rs} \int_0^t \lambda_r \lambda_s dt - \sum_{r}^{n} \int_0^t \lambda_r Q_r \, dt + F(A_1, \ldots, A_{2n}), \tag{5}$$

worin F eine willkürliche Funktion der Argumente A_r bedeutet.

Da die $\lambda_r(t)$ bekannt und die Systemkräfte gegeben sind, erhalten wir:

$$\int_0^t \lambda_r Q_r \, dt = D_r \sum_{\alpha=1}^{2n} a^{r\alpha} A_\alpha \left[\frac{e^{(i\varepsilon - \nu_\alpha)t} - 1}{i\varepsilon - \nu_\alpha} \right]. \tag{6}$$

Ersetzen wir die A_α in (6) durch ihre Werte [IV(4)], einführen die ausgerechneten Werte der Quadraturen (6) [IV(14)] in den Ausdruck für z (5) und setzen dann:

$$\lambda_r = \xi_r, \quad \mu_r = \xi_{r+n} \qquad (r = 1, \ldots, n),$$

es wird

$$z = \frac{1}{2i} \sum_{r,s}^{n} \sum_{\alpha,\beta}^{2n} \sum_{i,j}^{2n} b_{rs} a^{r\alpha} a^{s\beta} a_{i\alpha} a_{j\beta} \left[\frac{e^{(\nu_\alpha + \nu_\beta)t} - 1}{\nu_\alpha + \nu_\beta} \right] \xi_i \xi_j -$$

$$- \sum_{r}^{n} \sum_{\alpha}^{2n} \sum_{k}^{2n} D_r a^{r\alpha} a_{k\alpha} \left[\frac{e^{i\varepsilon t} - e^{\nu_\alpha t}}{i\varepsilon - \nu_\alpha} \right] \xi_k + F(A_1, \ldots, A_{2n}). \tag{7}$$

Setzen wir ferner $v_r = y_r$, $x_r = y_{r+n}$ $(r = 1, \ldots, n)$ und führen in [I(6)] anstelle von λ_r und μ_r ihre Ausdrücke [IV(12)] ein; dann folgt aus (5) und [I(6)] für $t = 0$:

$$F(A_1, \ldots, A_{2n}) =$$

$$= \lg \int_{-\infty}^{+\infty} \cdots \int e^{-\sum_{r,\alpha}^{2n} a^{r\alpha} A_\alpha y_r} f(y_1, \ldots, y_{2n}, 0) \, dy_1, \ldots dy_{2n}. \tag{8}$$

Den Ausdruck (8) für F setzen wir in (7) ein und da $e^z = \Phi$ ist, erhalten wir

$$\Phi(\xi_1, \ldots, \xi_{2n}, t) =$$

$$= e^{\sum_{ij}^{2n} N_{ij} \xi_i \xi_j - \sum_{k}^{2n} P_k \xi_k} \int_{-\infty}^{+\infty} \cdots \int e^{-\sum_{r}^{2n} d_r y_r} f(y_1, \ldots, y_{2n}, 0) \, dy_1 \ldots dy_{2n}, \tag{9}$$

worin

$$N_{ij} = \frac{1}{2a} \sum_{r,s}^{n} \sum_{\alpha,\beta}^{2n} b_{rs} a^{r\alpha} a^{s\beta} a_{i\alpha} a_{j\beta} \left[\frac{e^{(\nu_\alpha + \nu_\beta)t} - 1}{\nu_\alpha + \nu_\beta} \right], \tag{10}$$

$$P_h = \sum_{r}^{n} \sum_{\alpha}^{2n} D_r a^{r\alpha} a_{h\alpha} \left[\frac{e^{i\varepsilon t} - e^{\nu_\alpha t}}{i\varepsilon - \nu_\alpha} \right], \quad d_r = \sum_{\alpha}^{2n} a^{r\alpha} A_\alpha$$

ist.

Mit Φ von der Form (9), nachdem in (9) y durch y' ersetzt wird. bilden wir die Verteilungsfunktion f:

$$f(y_1, \ldots, y_{2n}, t) =$$

$$= \frac{1}{(2\pi i)^{2n}} I \int_{-\infty}^{+\infty} \ldots \int e^{-\sum_r^{2n} d_r y'_r} f(y'_1, \ldots, y'_{2n}, 0)\, dy'_1 \ldots dy'_{2n}, \qquad (11)$$

worin

$$I = \int_{-i\infty}^{+i\infty} \ldots \int e^{\sum_{ij}^{2n} N_{ij}\xi_i\xi_j + \sum_i^{2n} M_i\xi_i} d\xi_1 \ldots d\xi_{2n}, \quad M_i = y_i - P_i \qquad (12)$$

ist.

§ 3. Die Koeffizienten N_i und M_i im Exponent der Funktion unter dem Integralzeichen von (12) sind komplex imaginär, während ξ_i rein imaginär ist.

Um den Integralausdruck (12) zu berechnen, setzen wir

$$N_{ij} = p_{ij} + iq_{ij}, \quad M_i = y_i - h_i + il_i, \quad \xi_i = i\eta_i, \qquad (13)$$

dann nimmt das Exponent der Funktion unter dem Integralzeichen die Form:

$$-\sum_{ij}^{2n} (p_{ij}\eta_i\eta_j + iq_{ij}\eta_i\eta_j) - \sum_i^{2n} [l_i\eta_i + i(h_i - y_i)\eta_i]. \qquad (14)$$

Es seien die erste bzw. zweite reelle quadratische Form des Ausdrucks (14) durch φ_1 bzw. φ_2 bezeichnet.

Durch eine Orthogonaltransformation $\eta_i = \sum_j^{2n} c_{ij}\vartheta_j$ wird die quadratische Form φ_1 als $\varphi'_1 = \sum_k^{2n} \delta_k \vartheta_k^2$ dargestellt; die δ_k werden aus der Gleichung:

$$|p_{ij} - \delta_{ij}| = 0 \quad (i, j = 1, \ldots, 2n), \qquad (15)$$
$$(i \neq j, \quad p_{ij} - \delta_{ij} = p_{ij}, \; i = j, \; p_{ij} - \delta_{ij} = p_{ij} - \delta)$$

gefunden, während die Koeffizienten c_{ij} aus dem entsprechenden linearen System erhalten werden. Durch die Substitution $\vartheta_i = \frac{1}{\sqrt{\delta_i}}\zeta_i$ wird die Form φ'_1 zu einer Einheitsform reduziert. Die Form $\varphi_2(\eta)$ wird zu $\varphi'_2(\zeta)$.

Eine Orthogonaltransformation $\zeta_j = \sum_k^{2n} m_{jk} u_k$ gibt ferner der Form φ'_2 eine kanonische Gestalt $\varphi_2 = \sum_r^{2n} \gamma_r u_r^2$; die γ_r finden wir aus der Gleichung:

$$|q_{ij} - \gamma p_{ij}| = 0 \qquad (i, j = 1, \ldots, 2n), \qquad (16)$$

während die Koeffizienten m_{jk} aus dem entsprechenden linearen System erhalten werden. Lineare Formen ψ_1 und ψ_2 im Ausdruck (14) werden mit Koeffizienten

$$g_r = \sum_{ij}^{2n} \frac{l_i}{\sqrt{\delta_j}} c_{ij} m_{jr}, \qquad k_r = \sum_{ij}^{2n} \frac{(h_i - y_i)}{\sqrt{\delta_j}} c_{ij} m_{jr} \qquad (17)$$

behaftet. Das Integral (12) erhält, in neuen Veränderlichen ausgedrückt, die Gestalt

$$I = (i)^{2n} J \int_{-\infty}^{+\infty} \cdots \int e^{-\sum_{r}^{2n} [(1+i\gamma_r) u_r^2 + (g_r + ik_r) u_r]} du_1 \ldots du_{2n}, \qquad (18)$$

worin J die Jacobische Determinante der Transformation bedeutet:

$$J = \left| \sum_{s}^{2n} \frac{c_{is} m_{sj}}{\sqrt{\delta_s}} \right| \qquad (i, j = 1, \ldots, 2n). \qquad (19)$$

Durch die Substitution

$$u_r = \frac{1}{\sqrt{1+i\gamma_r}} z_r - \frac{g_r + ik_r}{2(1+i\gamma_r)}$$

wird dem Integral (18) eine leicht zu berechnende Gestalt gegeben; es wird

$$I = \frac{(i\sqrt{\pi})^{2n}}{\sqrt{\rho_1 \ldots \rho_{2n}}} J e^{\sum_r \tau_r^2}, \qquad (20)$$

worin

$$\rho_r = 1 + i\gamma_r,$$

$$\tau_r = \frac{1}{2\sqrt{1+i\gamma_r}} \sum_{i,j}^{2n} \left[\frac{l_i + i(h_i - y_i)}{\sqrt{\delta_j}} c_{ij} m_{jr} \right]$$

ist.

Folglich hat die Verteilungsfunktion, durch den Anfangszustand $(t = 0)$ ausgedrückt, in den Veränderlichen v_r und x_r die Form:

$$f(v_1, \ldots, v_n, x_1, \ldots, x_n, t) =$$

$$= \frac{1}{(2\sqrt{\pi})^{2n} \sqrt{\rho_1 \ldots \rho_{2n}}} J e^{\sum_r^{2n} \tau_r^2} \int_{-\infty}^{+\infty} \cdots \int e^{-\sum_r^n (d_r v_r' + d_{r+n} x_r')} \times \qquad (21)$$

$$\times g(v_1', \ldots, v_n', x_1', \ldots, x_n') \, dv_1' \ldots dv_n' \, dx_1' \ldots dx_n'$$

worin

$$g(v_1', \ldots, v_n', x_1', \ldots, x_n') = f(v_1', \ldots, v_n', x_1', \ldots, x_n', 0)$$

ist.

§ 4. Die Gleichung der Funktion z in der Theorie der Brownschen Bewegung hat die Form:

$$\frac{dz}{dt} = \frac{1}{2a} \sum_{r,s}^{n} b_{rs} \lambda_r \lambda_s - \sum_{r}^{n} \lambda_r Q_r. \qquad (22)$$

Wenn wir die «mechanischen Gleichungen» in der Lagrangeschen Form bilden, und dann die r-te Gleichung mit v_r multiplizieren und nach r summieren, so wird eine analoge Gleichung erhalten:

$$\frac{d}{dt}(T + U) = -\sum_{r,s}^{n} b_{rs} v_r v_s + \sum_{r}^{n} v_r Q_r. \qquad (23)$$

Auf diese Analogie wurde von V. A. Dmitrijev hingewiesen.

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GEOPHYSICS

AN ELEMENTARY THEORY OF THE SPRING SUSPENSION OF THE VERTICAL SEISMOGRAPH

By G. L. SCHNIRMAN

(Communicated by S. I. Wawilow, Member of the Academy, 1. X. 1937)

Let us examine the simplest form of vertical seismograph (Fig. 1) consisting of a mass suspended on a cylindrical spiral spring and having a horizontal axis of rotation. Its parameters are: M—mass; R—distance between axis of rotation to the centre of gravity; r—distance between the point of attachment of the lower end of the spring to the pendulum rod and the axis of rotation. The spring is considered to be imponderable.

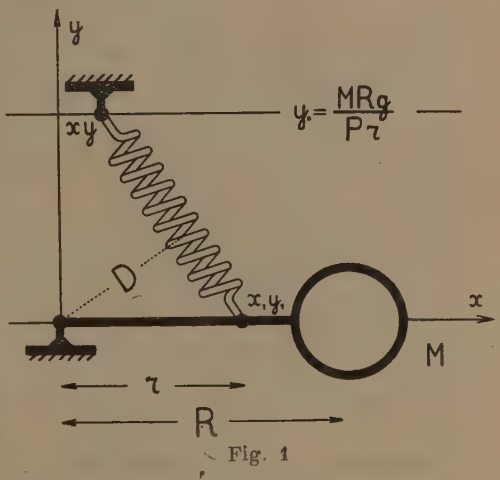

Fig. 1

The position of the point of attachment of the upper end of the spring to the foundation of the seismograph is so far left indeterminate. The other assumptions are: 1) the centre of gravity of the pendulum, the point of attachment of the spring to the rod and the axis of rotation lie in the same plane; 2) the spring has a hinged connection with the pendulum and the foundation of the seismograph; 3) no moments of elastic forces are set up by either the axis of rotation or the hinges of the spring suspension when the pendulum oscillates; 4) there are no frictional forces in this system.

We cause the origin of the rectangular system of coordinates to coincide with the axis of rotation of our seismograph in such a manner as to get the y axis into a vertical position, while the x axis lies in the plane of oscillation of the pendulum. In this case, whatever the position of the pendulum, a simple relation

$$x_1^2 + y_1^2 = r^2 \qquad (1)$$

will be found to exist between the x_1, y_1 coordinates of the point of attachment of the lower end of the spring to the pendulum rod and the distance r of that point to the axis of rotation of the pendulum, or, if the horizontal position of the pendulum is taken for the initial one, then

$$x_1 = r\cos\Theta \quad \text{and} \quad y_1 = r\sin\Theta \qquad (2)$$

where Θ is the angle of deflection of the pendulum.

If we denote by P the rigidity of the spring, by A its initial length in the absence of a force of deformation, and by z its full

length when deformed, the force developed by such a deformed spring would be

$$f = P(z - A). \tag{3}$$

The full length z of the spring may be determined, at any angle of deflection of the pendulum, as the distance between two points having the coordinates x_1, y_1 and x, y respectively

$$z = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}. \tag{4}$$

The force developed by the expanding spring is thus:

$$f = P\left[\sqrt{(x_1 - x)^2 + (y_1 - y)^2} - A\right]. \tag{5}$$

That force is applied to arm D (Fig. 1) equal to the distance between the axis of the spring (the line connecting the points of attachment of the spring) and the origin of the coordinates. This distance (analytically determined) is:

$$D = \frac{x_1 y - y_1 x}{\sqrt{(x_1 - x)^2 + (y_1 - y)^2}}. \tag{6}$$

From the expressions (5) and (6) obtained we determine the moment of elastic force of the spring about the axis of rotation of the pendulum:

$$\mathfrak{M}_1 = fD = P(x_1 y - y_1 x)\left[1 - \frac{A}{\sqrt{(x_1 - x)^2 + (y_1 - y)^2}}\right]. \tag{7}$$

We substitute in the expression obtained the values of x_1 and y_1 from (2) and simplify the radical expression:

$$\mathfrak{M}_1 = Pr(y\cos\Theta - x\sin\Theta)\left[1 - \frac{A}{\sqrt{r^2 + x^2 + y^2 - 2r(x\cos\Theta + y\sin\Theta}}\right]. \tag{8}$$

Before developing formula (8) any further, we shall consider a most interesting special case, which greatly contributes to the elucidation of the question.

Let be $A = 0$ in formula (8). The physical meaning of this expression is that the length of the non-deformed spring is zero, i. e. that, in the absence of a tensile force, the ends of the spring would coincide, if the neighbouring coils of the latter resting on each other when the spring is closed did not prevent this. We thus have:

$$\mathfrak{M}_{10} = Pr(y\cos\Theta - x\sin\Theta). \tag{9}$$

The moment of the force of gravity acting on our pendulum is

$$\mathfrak{M}_2 = -MRg\cos\Theta. \tag{10}$$

To ensure a reliable operation of the vertical seismograph, its centre of gravity should, so long as the latter is in equilibrium, lie in the same horizontal plane as the axis of rotation. It is necessary that, at $\Theta = 0$, the moment of the elastic force of the spring should balance the moment of the force of gravity, i. e.

$$\mathfrak{M}_{10} + \mathfrak{M}_2 = 0. \tag{11}$$

By summing up (9) and (10), substituting $\Theta = 0$ therein and equating the sum to zero, we obtain:

$$y_0 = \frac{MRg}{Pr}. \tag{12}$$

Thus, in the case of a pendulum with a null initial length of the spring ($A = 0$) and with prescribed values of parameters P, r, M and R, no horizontal position of static equilibrium can be achieved by the pendulum unless condition (12) is fulfilled, i. e. unless the upper point of the spring's suspension lies at arbitrary point of a straight line (Fig. 1) parallel to the x axis (i. e. a horizontal one) and determined by equation (12). Any variation in any parameter of the seismograph will naturally alter the height of our straight line above the x axis.

In our subsequent considerations it is assumed that the seismograph operates normally, i. e. that condition (12) is fulfilled.

We substitute (12) in (9), add to (10) and reduce; hence

$$\mathfrak{M}_{10} + \mathfrak{M}_2 = -Prx \sin \Theta. \tag{13}$$

The equation of motion of the seismograph in the absence of external and frictional forces may be written thus:

$$K\ddot{\Theta} + Prx \sin \Theta = 0 \tag{14}$$

where K is the moment of inertia of the pendulum about the axis of rotation. Considering the angular deflections of the pendulum to be small, we substitute Θ for $\sin \Theta$ and write the equation in the form:

$$\ddot{\Theta} + \frac{Prx}{K} \Theta = 0. \tag{15}$$

From equation (15), the period of the pendulum's natural oscillations is

$$T = 2\pi \sqrt{\frac{K}{Prx}}. \tag{16}$$

The formula obtained expresses in a very simple manner the relation between the period of the pendulum's natural oscillations, its parameters and the abscissa of the point of attachment of the upper end of the spring to the foundation of the seismograph. As the point of attachment of the upper end of the spring approaches the y axis, the period of the pendulum's natural oscillations increases, reaching infinity at $x = 0$*.

We now pass on to the question of the displacement in zero-point of the pendulum and the change of its period due to the thermal changes in the rigidity of the spring. In the case of a seismograph with a null initial length of spring, this question (a very essential one for a vertical seismograph) may be solved in a comparatively simple form.

Let us assume that, owing to a thermal variation in the rigidity of the spring equal to ΔP, the pendulum has been deflected from its initial horizontal position by the angle φ. If this new position of the pendulum is taken as the initial one and if the angles of rotation are measured from it, it is easy, by successive transformation of the above formulae, to obtain the following relations between the variations in the rigidity of the spring (ΔP), the angle of deflection of the pendulum (φ) and the initial (T_0) and the new (T_1) period of the natural oscillations of the pendulum:

$$\operatorname{tg} \varphi = \frac{T_0^2 g}{4\pi^2 l} \cdot \frac{\Delta P}{P + \Delta P}, \tag{17}$$

$$T_1 = T_0 \sqrt{\cos \varphi - \frac{4\pi^2 l}{T_0^2 g} \sin \varphi}, \tag{18}$$

$$T_1 = T_0 \frac{1}{\sqrt[4]{\frac{(P+\Delta P)^2}{P^2} + \frac{T_0^4 g^2}{16\pi^4 l^2} \cdot \frac{(\Delta P)^2}{P^2}}}. \tag{19}$$

Let us now revert to the general case of the pendulum with a spring of an arbitrary initial length. Such a spring may be imagined as a spring of null initial length, connected to an extension rod, the length of which is equal to the initial length of the spring. If the spring had no initial

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* An absolutely analogous result in the special case of $x = 0$, was obtained by Lucien La Coste (A New Type Long Period Vertical Seismograph, Physics, July 1934, Vol. 5, N 7, p. 178—180).

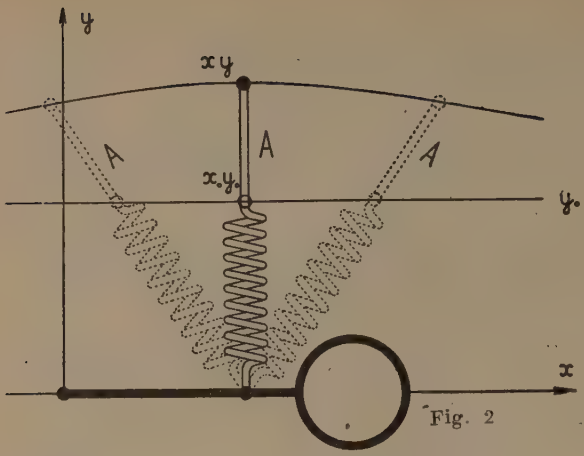

Fig. 2

length, its upper end could be displaced along the straight line y_0 (Fig. 2) without disturbing thereby the position of static equilibrium. In our case, however, a spring of null initial length has been lengthened at its upper end by a non-extensible rod, and it is natural that, in order to reach the horizontal position of static equilibrium, the upper point of the spring of null initial length should not deviate from the straight line y_0, while the upper point of suspension (the upper end of rod A), would thereby be displaced by the value A away from the lower point of suspension of the spring. At $x = r$, the trajectory of that point is at distance A from the straight line y_0 and approaches it asymptotically when x is displaced into plus or minus infinity.

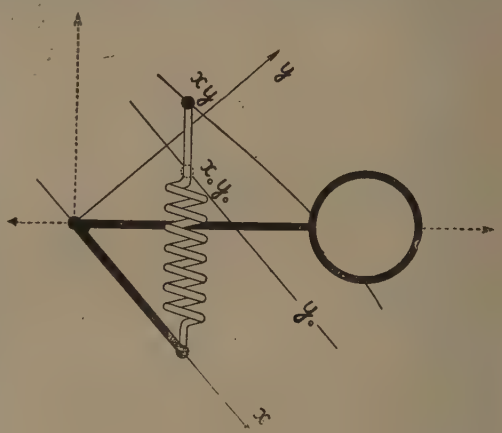

Fig. 3

In this case, it is very difficult to obtain an expression for the pendulum's natural oscillations. The problem is greatly simplified if, instead of the parameters x and y (the coordinates of the upper point of suspension of the spring), the x_0 and y_0 (coordinates of the upper point of the imaginary spring of null initial length) are taken to be the fundamental ones. In this case we obtain a rather cumbersome, but quite an exhaustive relation:

$$T = 2\pi \sqrt{\frac{K}{Pr\left\{x_0 - \frac{r}{\left(1+\frac{1}{A}\sqrt{(r-x_0)^2+y_0^2}\right)\left[1+\frac{y_0^2}{(r-x_0)^2}\right]}\right\}}}, \qquad (20)$$

where y_0 is determined by formula (12).

All the above considerations, as can easily be shown, hold good not only in the case of the axis of rotation of the pendulum, its centre of gravity and the point of attachment of the spring to the pendulum lying in the same plane. They hold good for any position of the point of attachment of the spring to the pendulum, the only conditions being that our rectangular system of coordinates should be rotated in such a manner that the x axis should cross the point of attachment of the spring to the pendulum (Fig. 3).

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PHYSICAL CHEMISTRY

CORROSION OF IRON AT THE BOUNDARY: METAL-LIQUID-GAS AND METAL-LIQUID-LIQUID

By M. K. TIKHONOV

(Communicated by V. A. Kistiakovskij, Member of the Academy, 31. VII. 1937)

In the works of V. A. Kistiakovskij, Evans, Liebreich, Maass and other students of corrosion, the accelerated methods of corrosion testing of metals occupy an outstanding place as they are of great theoretical and practical importance. These methods comprise also that of drop corrosion, the mechanism of which is a point of discussion among a number of scientific workers.

What are the most important facts to be noted in the study of corrosion in a drop of an electrolyte, is not yet clear. Evans (1) explains the corrosion process in a drop of an electrolyte by the ununiform aeration (uneven access of oxygen), i. e. the oxygen of the air more easily reaches the edges of the drop than its centre. Therefore, the periphery of the drop is the cathode while its interior part becomes the anode and begins to dissolve with the formation of iron salt. The ferrous hydroxide, which is formed in the centre of the drop, under the action of the oxygen of the air, oxidizes to brown ferric hydroxide. The latter is distributed as a membrane of rust along the periphery of the drop or close to it. Academician V. A. Kistiakovskij (2, 3) without denying the importance of differential aeration also holds that in drop corrosion of iron the corrosion products are transferred to the periphery of the drop cataphoretically. The mechanism of development of the membrane of rust is described in the work of M. K. Tikhonov (4). Maass and Liebreich (5) do not attach great importance to the variation in the concentrations of oxygen and explain the enhanced corrosion in the centre of the drop by the fact that the convection currents passing through it deposit hydroxides, i. e. the products of the corrosion which has already taking place, in the centre of the drop, these corrosion products accelerating the course of the process.

Academician V. A. Kistiakovskij (2) states that «it is difficult to explain the process of corrosion in a drop of an electrolyte only by the depolarizing influence of oxygen. Concentration of the ions of hydrogen has, apparently, a greater quantitative influence here. The difference in concentration of the ions of hydrogen in a film of moisture covering fissures is, in our opinion, responsible for the enhancement of corrosion in the fissures. It can be seen, indeed, that a negligible amount of rust may rapidly develop in these fissures which is, however, sufficient for communicating a weak alkaline reaction to the adsorption layer of the moisture. At that moment an acid reaction takes place on the surface of the films of the outer layer of the iron, which is associated with the presence of dissolved carbonic acid gas».

The object of the present work is to analyse a case of corrosion in a drop of an electrolyte and to determine the mechanism of the process of drop corrosion both in atmospheric conditions and at the boundary of metal-liquid-liquid with a corroding agent which hitherto has not been applied in the testing of coatings of ferrous metals by the drop method, viz. with a saturated solution of corrosive sublimate. If a drop of such solution is applied to the surface of polished iron or steel there begins at once a process of corrosion, accompanied by formation of minute droplets of mercury on the entire surface which gradually increase. The corrosion results in the formation, both at the periphery and in the centre of the drop, of a film of hydroxide the layers of which come off in some places. Neither cathode nor anode zones which are formed in drop corrosion, e. g. in the case of a 3% solution of NaCl, have been observed in the experiments in question. If, however, the drop is washed out within 15—20 minutes after the beginning of the experiment, corrosion centres nuclei can be seen, by means of a microscope, to be formed along the entire surface under the drop in the most susceptible sections of the film ([6]). The intensified process of corrosion is accounted for by the formation of a great number of local elements between the sections of the metal, which become anodes, and the droplets of mercury formed on the surface of the metal (as a result of the phenomena similar to amalgamation of iron), which play the role of cathodes. Thus the whole surface under the drop represents a mass of local elements upon the work of which the differential aeration of the oxygen of the air does not exert a visible influence. On Fig. 1 is shown the surface of the plate after the experiment.

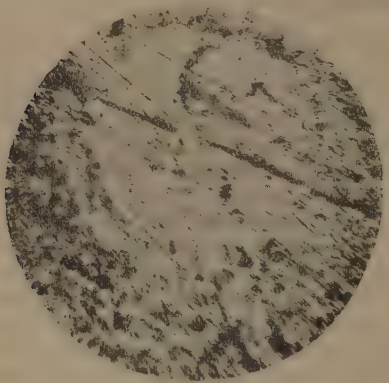

Fig. 1.

The second series of experiments were performed at the boundary metal-liquid-liquid. Distilled benzene was used as the second liquid. In these experiments the corrosion process proceeds in a different way than in those under atmospheric conditions. Immediately after application of a drop of sublimate solution, corrosion begins not along the entire surface but essentially in the centre of the drop. In the centre of the drop minute droplets of mercury begin to appear which are gradually aggregated and this tends to intensify the process of corrosion. The process of corrosion is accompanied by an under microscope (magnifying power 100) hardly perceptible liberation of bubbles of hydrogen which at the moment of liberation collide and, consequently, small bubbles unite into larger ones which are capable of bursting the coat of the drop. Its bursting can be easily demonstrated in the following way: around the drop, under the layer of benzene there are formed numerous minute centres of corrosion which eventually spread throughout the surface of the plate of metal not covered by the drop of sublimate solution. The formation of a series of fresh corrosion centres may, apparently, be accounted for by the fact that the films of the bubbles of hydrogen are at the moment of their formation saturated with the sublimate solution, i. e. a solvate coat is formed which makes the bubble heavier and prevents its being liberated through the layer of benzene. Such a bubble is consequently deposited on the metal, it bursts, and the resulting drop of the sublimate solution produces corrosion. A microscopic examination of the corrosion centres situated in a fan-like

way and close to the drop clearly shows that the formation of corrosion centres is caused by the corrosive sublimate as droplets of mercury are well visible. If within 20—25 minutes after the beginning of the experiment the drop is washed out and the surface covered by the drop is examined with a microscope, it is found that the centre of the drop is more corroded than the periphery, a great number of deep corrosion centres being present in the middle of the drop. Fig. 2 shows the surface of a plate of iron within 20 minutes after the beginning of the experiment. Not only the corrosion centres in the centre of the drop may be seen on the figure, but also a number or corrosion centres near the drop. As the access of the oxygen of the air, through the layer of benzene, was more difficult towards the centre of the drop of sublimate solution, the corrosion process proceeded essentially in the centre of the drop which is in accord with Evans differential aeration theory.

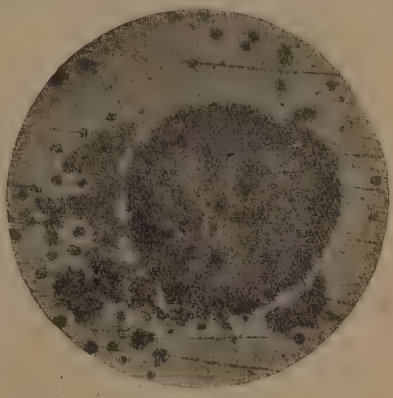

Fig. 2.

D. I. Mirlis ([7]) studying corrosion under the same conditions as the author, but using a solution of muriatic acid as an electrolyte, came to the conclusion that «corrosion of murials under the conditions of polyphase contact metal-liquid-liquid, metal-liquid-gas is of linear character» which does not agree with Evans differential aeration theory and the experiments described in this paper. It seems to the author that the process of corrosion in a drop of an electrolyte depends on the choice of the electrolyte and the conditions of experiment.

Conclusions

1. A solution of corrosive sublimate can be applied as a strong corroding agent in the study of the resistance properties of oxide-adsorption films by the drop method.

2. Corrosion of iron in a drop of sublimate solution under atmospheric conditions proceeds in the same degree both in the centre and at the periphery of the drop.

3. Corrosion of iron in a drop of sublimate solution at the metal-liquid-liquid boundary proceeds essentially in the centre of the drop which is in accord with Evans differential aeration theory.

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ZOOLOGY

ON THE NUTRITION OF SOME *CYCLOPIDAE (CRUSTACEA)*

By N. A. DZIUBAN

(Communicated by S. Zernov, Member of the Academy, 17. IX. 1937)

In the present paper we state some of the results of our observations on the nutrition of *Cyclops viridis*, *C. insignis*, *C. vernalis* and *C. albidus**.

The subject of the present paper forms part of the theme which is being studied by the Department of Hydrobiology of the Moscow Fishery Institute under the guidance of Prof. N. S. Gaevskaja (viz. the food interrelationships in water basins). The material for this paper was collected in the ponds of Petrovsko-Razumovskoe (Moscow). The food was studied both by means of an analysis of the contents of the intestine of individuals taken directly from the ponds and by means of experiments. In the first case, the animals were preliminarily washed in distilled water, and then the contents of the intestine were squeezed out by carefully pressing the anterior part of the body with a needle through a cover glass. 60 such dissections were made, 15 for each species. In some cases, the entire intestine was successfully extracted by dissecting the cyclops under a binocular magnifier. In all cases the contents of the intestine was an amorphous yellowish mass with a slight greenish tint. No animal or plant remains were present, excepting one case, where a specimen of a small *Navicula* was found. These data might at first sight appear to justify the conclusion that the afore-named cyclop species feed on detritus; direct observations, however, enabled us to establish the true nature of this food mass, as well as of the nutrition of these four species.

The cyclops were first given plant food (*Protococcales*, minute *Diatomaceae*, *Euglena*, as well as minute *Bivularia* and *Anabena*). An analysis of the intestinal contents of 28 individuals showed that the above-named cyclops species do not take this food at all.

Daphnia pulex, *D. magna*, *Copepodites*, *Planaria* and *Oligochaeta* were offered them as animal food. The cyclops in question immediately revealed their carnivorous nature. They ate quite readily all the above-named animals. It should be added that cyclops do not pursue their prey. They only capture it when they collide with it, or when their victim is at a distance of 1—2 mm from them, either below or to one side of them. The cyclop greedily seizes its victim with a quick jerk and after two or three violent sideway motions sinks with its victim to the bottom of the vessel, where it remains motionless. The vessel may be moved during that time, the cyclops themselves may be pushed to the walls of the vessel or even turned

* The literature contains but fragmentary data on the nutrition of cyclops. Up to the present this problem has not been specially studied.

over with a needle for more convenient observation. The cyclops hardly ever let go their prey during these operations.

The amounts of daphnia consumed during 24 hours by the four above-mentioned cyclop species are about equal, but the mode of consumption is different. *C. albidus* sucks up, as it were, the daphnia: one may observe how the tissues of the daphnia pass into the intestine of the cyclop, which grows gradually darker, whilst the daphnia becomes transparent. In a few minutes, the cyclop cast the entirely empty skin away. The other three species, *C. viridis*, *C. insignis* and *C. vernalis*, consume the entire daphnia. The cyclops usually start to eat the daphnia from the ventral side of the upper part of its body. If the cyclop cathches the daphnia by the other end, it always turns its victim round so as to clutch it with its jaws in the above-indicated position. We have established that the maximum size of daphnias a cyclop can seize is 1.01—1.30 mm. The manner of eating daphnias 0.5—0.6 mm in size, markedly differs from that of eating daphnias close in length to the limit the cyclop can manage. When cyclops are fed with daphnia 0.5—0.6 mm long, they usually show great voracity during the first period of the test (the number of daphnias remains constant throughout the test). During the first 30—40 minutes after the introduction of the daphnias, a cyclop would eat 3—4 individuals, but after that the rate of consumption began to fall. Nevertheless, the number of specimens of such size consumed per day was fairly high, ranging from 10 to 14 per one cyclop, not subjected to preliminary fasting. Incompletely consumed daphnia were frequently found in the vessel. But when specimens 1.01—1.30 mm

Table 1

Consumption of Daphnias by Cyclops

Series	Number of cyclops	Number of daphnias introduced	Size of daphnias introduced (mm)	Number of daphnias consumed per day	Number of cases in 20 tests
I	1	5	1.03—1.08	2	20
II	1	8	1.03—1.20	2	18
				1	2
III	1	12	1.01—1.14	2	20
IV	1	16	1.10—1.30	2	17
				1	3
V	2	4	1.07—1.29	4	19
				3	1
VI	2	8	1.07—1.29	4	19
				2	1
VII	2	12	1.12—1.27	3—4	20
VIII	2	16	1.08—1.25	2	20
IX	4	5	1.17—1.25	5	20
X	4	8	1.18—1.27	6	1
				7	2
				8	17
XI	4	12	1.13—1.30	7	3
				9	2
				8	15
XII	4	16	1.12—1.27	7	3
				8	16
				9	1

long are fed to the cyclops, the results are quite different. The cyclop catches daphnias at intervals of several hours, and it eats up (or sucks up in the case of *C. albidus*) its prey very thoroughly. It should be added that when specimens of various sizes were introduced into the vessel containing the cyclops the latter inevitably began by attacking the smaller individuals. If the quantity of daphnias introduced into the vessel was not supplemented, the cyclops would in time consume all of them. We carried out some tests for the purpose of determining the diurnal consumption of daphnias by cyclops at different densities of the population of both the ones and the others in 75 cm³ of water. A number of different combinations was tested, from 1 to 4 cyclops and from 4 to 16 daphnias. These tests showed that the diurnal consumption of daphnias remained close to 2, as may be seen from Table 1, which contains for lack of space, only part of our investigations.

C. insignis, *C. viridis*, *C. vernalis* and *C. albidus* readily feed on small copepod larvae of a length of about 1 mm. On colliding with a copepod larva or on approaching it to a distance of 1—2 mm, the cyclop pounces upon the larva with a sudden jerk and seizes it, usually in a such a way that the larva's furca is turned towards the cyclop's oral parts. Then it sinks with its prey to the bottom of the vessel. As in the case of daphnias, during this time the cyclop does not let its prey even if is turned over with a needle. From time to time the cyclop would turn and move about its prey with its swimming legs. In 5—7 minutes the copepod larva is entirely consumed. Frequently, on eating one larva, the cyclop attacks a second, which it would then leave not fully consumed. If no food were added, the cyclop would in a certain period of time eat up all the larvae present. We used in our tests different combinations of various densities of population both of the cyclops and the copepod larvae (viz. from 1 to 10 of the first and from 2 to 15 of the latter). Within these limits, the amount of larvae consumed by the cyclops varied but little, being on an average two per cyclop per day. This amount is identical with that for daphnia 1.01 to 1.30 mm long.

Table 2

Consumption of Copepod Larvae by Cyclops

Series	Number of cyclops	Number of larvae introduced	Number of copepod larvae consumed per day	Number of cases in 25 tests	Series	Number of cyclops	Number of larvae introduced	Number of copepod larvae consumed per day	Number of cases in 25 tests
I	2	2	1	4	X	3	6	6	18
			2	21				5	4
II	1	6	3	3	XI	3	10	3	2
			2	22				6	23
III	1	10	2	24	XII	3	15	5—6	25
			0	1	XIII	4	6	6	25
IV	1	15	2	23	XIV	4	10	8	21
			1	2				6	4
V	2	2	2	25	XV	4	15	7—8	25
VI	2	4	3—4	25	XVI	6	10	10	25
VII	2	6	3	5	XVI	6	15	11	3
			4	20				13	22
VIII	2	10	3—4	25	XVII	10	15	15	25
IX	2	15	3	2	XVII	10	20	18	2
			2	3				20	23
			4	20					

The greatest voracity was manifested by the cyclops when fed with planarians, especially in vessels containing no mud. The length of the planarians was from 4 to 6 mm. The cyclops caught very quickly the planarians introduced into the vessel. It was most interesting to observe the cyclops feeding on planarians. The planarian, which is several times as long as the cyclop, on being caught by the latter, wriggles energetically in its attempts to free itself. All such efforts are usually futile. Sometimes the planarian caught by one cyclop is also seized by another. Then a regular fight between the two cyclops ensues. They let loose the planarian and attack each other. The beaten combatant, sometimes wielding its extremities with difficulty, clumsily swims away.

A cyclop which has been fasting before the test, consumes a planarian 4—6 mm long in 15—20 minutes. It frequently happens that a cyclop eats only part of a planarian and later on finishes it up. The diurnal consumption of planarians by the above-mentioned cyclops species ranges from one to six per cyclop (Table 3). If the bottom of the vessel is covered with mud, into which the planarians bury themselves at times and then reappear again, the number of planarians consumed by the cyclops diminishes. Besides, in such cases the planarians are not completely eated up.

If the planarian population in vessels containing mud was very dense, the diurnal consumption per cyclop was the same as in vessels containing no mud.

Table 3

Consumption of Planarians by Cyclops

Series	Number of cyclops	Number of planarians introduced	Number of planarians consumed	Number of cases in 25 tests
I	1	2	2	25
II	1	4	1	4
			2	18
			4	3
III	1	8	2	10
			4	8
			5	5
			6	2
IV	2	2	2	25
V	2	4	4	25
VI	2	8	3	1
			4	10
			5	6
			6	8
VII	4	4	4	25
VIII	4	8	4	2
			8	23
IX	8	8	8	25

When the cyclops were given a mixed food consisting of planarians, daphnias and copepod larvae, they ate all these animals, but favoured most the planarians; next came the daphnias and then the larvae.

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PLANT PHYSIOLOGY

PHOTOSYNTHESIS AS APPLIED TO CITROUS PLANTS IN NATURAL CONDITIONS

By I. A. FILIPPENKO, E. H. GERBER and O. K. ELPIDINA

(Communicated by A. A. Richter, Member of the Academy, 14. IX. 1937)

The fact that citrous plants as well as many species of plants used in industry are widely cultivated in the moist subtropical region of the USSR has brought to the fore the problem of an exhaustive study of their biology. Up to quite lately, the study and control of the air nutrition of these plants had not attracted much attention on the part of the scientific research institutions of the Union. Numerous papers dealing with photosynthesis have been now accumulated in literature. A great wealth of evidence on photosynthesis has been fully analysed as to diurnal variations, as to the influence of geographical latitude and seasonal climatic changes, on the strength of a thorough investigation of a large number of vegetable species, citrous plants being, however, consequently absent therefrom.

The ecological study of photosynthesis will help to clear up the intricate biological complex involved in the relationship of plant and medium, and will thus hasten the solution of problems connected with a rational organization of citrous plant growing.

In 1935 and 1936, some members of the Institute of Plant Physiology, headed by academician A. A. Richter, carried out in the Batumi Botanical Gardens the photosynthesis of citrous plants and of some species used for industrial purposes.

The preliminary work done during the autumn months of 1935 by I. A. Filippenko and O. K. Elpidina, of this Institute, had ascertained that the various citrous species show a different response to light in respect of the photosynthesis energy and the height of the compensation points.The lemon and, to a less extent, the orange had proved to possess a greater need of light than the less exacting tangerine.

From June 10 to December 18, 1936, photosynthesis was also carried out in the Batumi Botanical Gardens by a group of workers of this Institute (I. A. Filippenko, E. H. Gerber and O. K. Elpidina). The object of their work was to study the dynamics of photosynthesis as affected by changes in strength of light and in temperature on a great variety of citrous species and of plants used in industry; also to ascertain the influence of natural seasonal climatic variations on the photosynthetic process in citrous plants. It was expected that the results obtained would enable the above workers to judge in how far it was expedient, from the point of view of the physio-

logy of air nutrition, to apply a dense growing of citrous plants, at the rate from 600 to 1000 trees per hectare.

I. Method of Work

In their pothosynthetic experiments, the research workers divided the joint effects of temperature and light, and endeavoured to embrace in their experiments all the light and temperature ranges characteristic of the seasonal conditions prevailing at Batumi. The experiments were made on cut leaves, which had been detached from the tree at stated morning hours. This was done in order to remove every possible doubt as to the knowledge of the preceding history of the leaf in question, a matter formerly mentioned by S. P. Kostyčev(1). The petiole of the leaf was sealed into one end of a three-way tube with a specially prepared paste, another end of the tube was sealed but its third end remained open to admit water, the petiole being thus supplied with water throughout the experiment. Atmospheric air was blown in through a glass chamber at a rate of 35 litres per hour. In order to maintain a constant temperature during the experiments, the chambers were immersed in water of a suitable temperature. In those experiments where a lower temperature was required, the air to be supplied to the chamber was first cooled in copper coils immersed in a cooling mixture. The CO_2 trap had been designed upon the principle described by Academician A. A. Richter(2). In calculating the difference in the titration between air control and experiment this difference was reduced to one hour and one square metre of leaf surface; the intensity of illumination was measured by a photoelectric cell graduated according to the readings of the galvanometer, with a subsequent recalculation into luxage. Variations of the intensity of illumination were achieved in the course of the experiments by screening off sunlight with a varying number of gauze layers. Humidity and air temperature were also recorded .The experiments were made in day time, from 11 a. m. to 3 p. m.

The summer and autumn of 1936 at Batumi were characterized by an abundance of atmospheric precipitations, cloudiness and humidity of air. The intensity of illumination was found to have decreased from 35 000 lux in July to 11 000 lux in December.

During the summer period from June to September, the experimenters recorded the dynamics of photosynthesis as affected by changes in the intensity of illumination up to the compensation point in 32 species of citrous plants and plants used in industry, at temperatures ranging from 25 to 30° i. e. characteristic of summer. From October to December, photosynthesis was studied in its relation to light variations at temperatures of 4—6°, 10—15° and also 20—25° for some of the above plants. These three ranges of temperatures were fairly characteristic of temperature conditions in autumn and early winter. The plants to be used in the experiments were partly taken from the Batumi Botanical Gardens and partly supplied by collective farms and peasant homesteads.

II. Photosynthesis in the Lemon Under Summer Conditions

The results obtained by photosynthetic experiments on the lemon are shown by curves in Fig. 1, the intensity of illumination (in lux) being plotted on the axis of the abscissae against the amount of assimilated CO_2 (in milligrams) in terms of 1 square metre per hour. These curves show a regularity common to all the lemons, namely, given a normal content of CO_2

in the air (0.450—0.500 mg CO_2 per litre of air) the intensity of illumination which at day temperatures of 25—28° reached on an average 35 000 lux at noon, resulted in a lowered photosynthesis in practically all the lemons. The maximum of photosynthesis corresponds to a range of 8500 to 12 000 lux, i. e. that luminous optimum is one-third of the summer noon sun.

The same relation to the direct source of light is shown by tangerines, oranges and by plants used for industrial purposes (tea, tung tree, cinchona).

Stocker (4), who had studied tropical plants from the point of view of the utilization by the latter of optical energy for processes of air nutrition in natural conditions, had obtained identical results.

In the autumn period (September—October), when the intensity of illumination in the Batumi district varies within the range of 15 000—

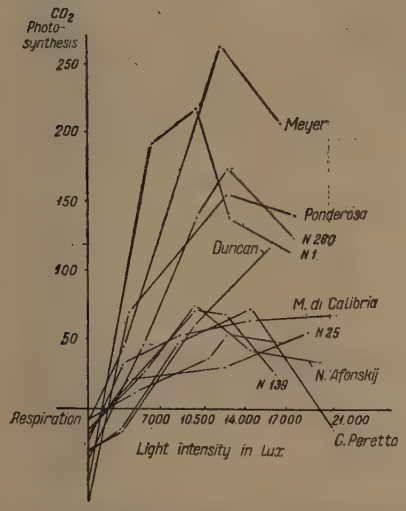

Fig. 1. Photosynthesis in some forms of lemon at different intensity of illumination

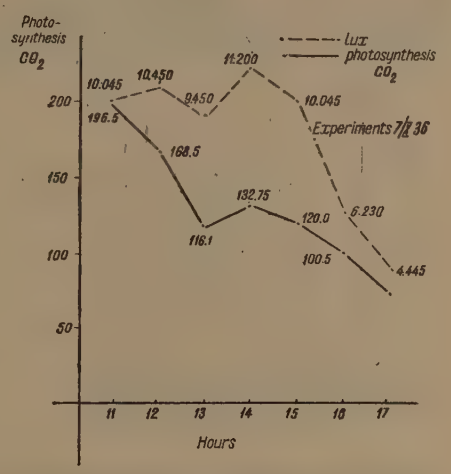

Fig. 2. Diurnal variations of photosynthesis in Novoafonski lemon

12 000 lux and when the prevailing day temperature is about 20° C, with ample atmospheric precipitations, the light factor is more fully utilized by the plants. This is clearly shown by the curve (Fig. 2) of the diurnal variations of photosynthesis in the Novoafonski lemon. The hours of the day are plotted on the axis of the abscissae against the amount of assimilated CO_2 in milligrams in terms of 1 square metre of leaf surface per hour, the dotted line showing the intensity of illumination in lux. Throughout the day, photosynthesis stood high, the curve showing no troughs. Given a favourable temperature and other conditions, the lowered autumnal luxage thus contributes to an enhanced photosynthetic activity of the leaf, and this produces a luxuriant autumnal growth of citrous plants; in the Batumi experiments, all the leaves showed a vigorous growth, which began at the end of August and lasted up to and including October. The steady photosynthesis during the day shows a great productivity of the leaf, the absence of superheating or overfilling of the leaves with assimilates. The synthetic activity of the vegetable organism is combined with the rapidity of its growth, a fact also observed by Kostyčev (3) in his photosynthetic experiments beyond the arctic circle.

III. Photosynthesis in Citrous Plants in the Autumn-Winter Period at Temperature of 4—6°, 12—15° and 23—25°

Photosynthetic experiments within these three temperature ranges and under varying optical conditions were made from the middle of October to the 18th of December. These investigations have proved the total absence of the so-called «quiescence period» in citrous plants during the winter. A great adaptability of these plants to seasonal meteorological conditions has been displayed by these experiments. This adaptability may be outlined as follows.

With the advent of winter, the optimum of illumination contributing to the maximum energy of photosynthesis is shifted towards a weaker illumination (Fig. 3), as shown by the experiments with Novoafonski lemon. As may be seen from that figure, the continuous line representing the progress of

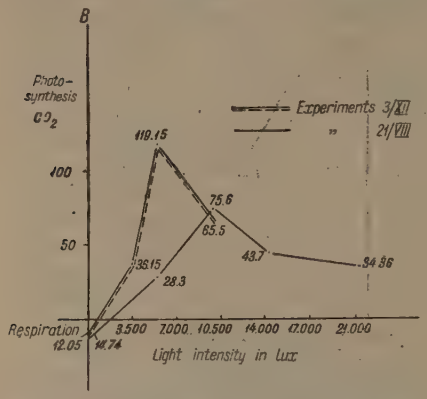

Fig. 3. Photosynthesis in Novoafonski lemon under different seasonal conditions

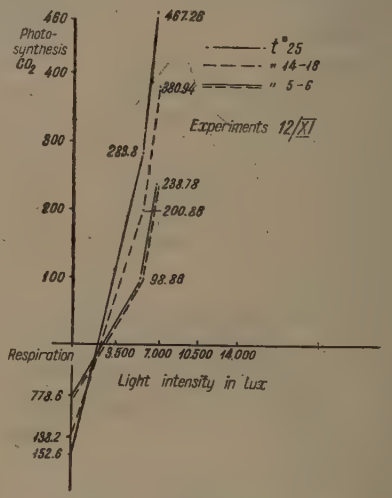

Fig. 4. Photosynthesis in the hybrid No. 172 at different intensity of illumination and temperature

photosynthesis in the experiments of August 21, at a temperature of 25°, shows the optimum of illumination to lie within the range of 10 000 lux; the optimum of illumination ascertained by the experiments of December 3, at a temperature of 14—15°, and shown by a broken line, lies within a range of 5000 lux.

The photosynthetic activity of the leaf is favourably affected by the fairly low diurnal temperatures characteristic of the autumn—winter period at Batumi, i. e. from 3 to 6° (Fig. 4, experiments of November 12, lemon hybrid № 172 and Fig. 5 of November 9, Klementin tangerine).

It may be seen from these figures that a lowering of temperature results in a diminished energy of photosynthesis and a diminished respiration. In some cases, the energy of photosynthesis may be at nearly the same level at a temperature of 3—6° as it is at 25°, a temperature characteristic of the summer months. This is clearly to be seen in Fig. 6 showing experiments with the lemon (mother of hybrid № 172) of November 15, where, at temperatures of 3—6, 13 and 25°, the maximum of photosynthesis have values of the same order.

Stocker (4), while working in the tropics with *Cassia* observed a fairly high increase of assimilation at temperatures of +5° and even lower. This led him to the conclusion that the temperature factor is utilized by the plant for its photosynthetic activity to a greater extent than the optical factor.

IV. Conclusions

The following conclusions may be drawn from the experimental evidence summarized above.

1. The photosynthetic experiments made with a different intensity of illumination and with a normal CO_2 content in the air, on various species of citrous plants, show that, in the summer months, an overwhelming majority of citrous and other plants do not require any direct solar illumination; the optical optimum favouring the maximum photosynthetic activity would appear to lie, for nearly

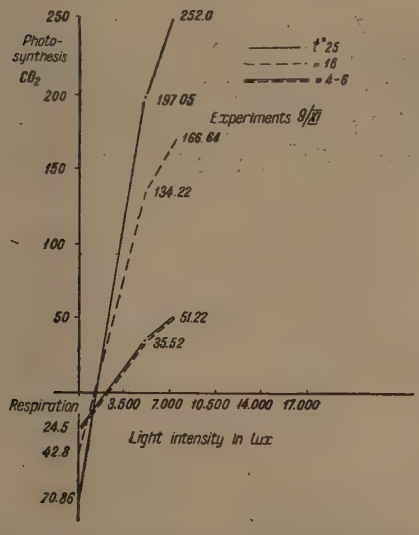

Fig. 5. Photosynthesis in the Klementin tangerine at different intensity of illumination and temperature

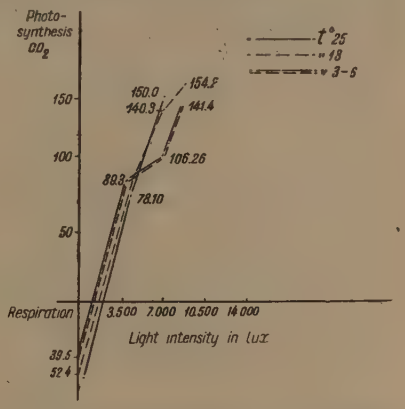

Fig. 6. Photosynthesis in the lemon (mother of the hybrid No 172) at different intensity of illumination and temperature

all the plants, within the range of 8500 to 12 000 lux, while, in the summer months, the intensity of illumination may be as high as 35 000 lux.

2. The above facts relating to the physiology of air nutrition of citrous plants prove that it is quite possible to grow such plants at an increased rate of 1000 (instead of 500) trees per hectare, provided the trees are properly pruned and the crowns are properly developed, and account is taken of the humidity of the climate, the nature of the soil and the relief of the land surface.

3. Citrous plants have no «quiescence period» in winter, and they possess a high adaptability to external conditions. As may be seen from the experiments, these plants show their adaptability by a shifting of their optimum of illumination towards a lower intensity of illumination, by adjusting their photosynthetic activity to lower temperature and, in some cases, even by a greater energy of photosynthesis than is observable in summer.

4. In view of the above, it is necessary, when sheltering citrous plants against winter frosts, to supply them periodically with such a winter optimum of light as would ensure the maximum photosynthetic process, so as

to protect the plants from starvation, a not unusual condition when they are under cover.

Agricultural practice will, as before, apply various kinds of covers to protect citrous plants from early autumn frosts. The high physiological activity of citrous plants, which has now been fully proved, and their notable adaptability to winter conditions makes it imperative to develop without delay a more definite method of ascertaining the best conditions of illumination etc. demanded by citrous plants kept under cover in winter time. The same conclusions are suggested by F. M. Zorin's (5) note on the response of citrous plants to light.

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PLANT PHYSIOLOGY

FORMATION OF BIOS IN THE YAROVIZED EMBRYOS OF WINTER WHEAT

By I. A. FILIPPENKO

(Communicated by A. A. Richter, Member of the Academy, 14. IX. 1937)

An insufficient knowledge of those physiological processes in plants which constitute their yarovization, compels physiologists to investigate these processes from different angles.

New lines of thought have lately appeared in the study of the physiology of development. N. G. Cholodnij ([1], [2]) advances the opinion that the qualitative changes in yarovized embryos are due to a change in the concentration of auxins or growth hormones. He assumes an increase in the concentration of growth hormone in the growing points of the embryo during yarovization and considers that this «must be the cause of the meristem cells of the young plant passing more quickly the first phases of development».

M. Kh. Čailachjan ([3],[4]), comes to the conclusion that the processes of the development of plants are associated with the regulatory activity of specific substances or hormones of flowering.

There are many papers which elucidate with sufficient clarity the biological role of different biocatalyzers in plant life. In their investigations of a number of plant forms in respect of bios content, N. Nielsen([5]), F. Kögl([6]), Boas ([7]), J. Dagys ([10],[11]), K. T. Sukhorukov ([8]) and D. Kliačko([9]) came to the conclusion that bios, which is not a specific factor of yeast multiplication only, is widely distributed in the plant and animal world.

Dagys ([10],[11]) says that the knowledge of the role of growth substances (bios) in the ontogenetic development of the higher plants, in the transformation and modification of matter, is still inadequate. His investigations of the bios content([11]) in buds and leaves of woody plants under various seasonal conditions and of its ditribution in maize seeds and seedlings ([10]), bring us right up to the problem of elucidating the role of bios in the development of the higher plants.

As early as 1936, during investigations of the physiological characteristics of yarovized and unyarovized winter wheats in the Laboratory of the Physiology of Development, at the Institute of Plant Physiology, Academy of Sciences of the USSR, several experiments were made for the purpose of determining the extent to which the yarovization processes occurring in the embryo are associated with the formation and accumulation of bios, as a hormone of cell division.

Determination of the bios content were made in seedlings of two winter wheat varieties Kooperatorka and Lutescens 0329, whose seeds had been

yarovized at temperature of from 0 to 3° C for different numbers of days. The embryos used in this experiment were grown both from completely and incompletely yarovized seeds.

The method of quantitative determination of bios, elaborated by E. Kling and A. Werner ([12]), was based on stimulating the growth of yeast by the addition of an extract of seed embryos to Haiduk's medium (sugar-asparagine-mineral medium).

The material was prepared in the following manner: the seeds were kept at room temperature, 15—20° C, for 4—5 days. The embryos obtained were then separated from the endosperm. 1—2 g samples of these embryos were then ground in a porcelain mortar, mixed with 40 cm^3 of distilled water and transferred to a 100 cm^3 flask. Sulphuric ether was then added, and the flask was placed in a thermostat for 24 hours at 37° C for autolysis. At the end of the 24 hours, the ether was removed, and the autolyzate was centrifuged for the purpose of separating the solid part. The liquid obtained was sterilized in an autoclave for 30 minutes at 110° C, and then added, in the amounts shown in the tables, to Haiduk's medium (25 cm^3) containing yeast. After 40 hours of yeast growth at 30° C, determinations of the number of yeast cells were made by means of the Thomas-Zeiss counting chamber. The result of the determinations are tabulated below (Tables 1, 2, 3, 4). The values in the tables are the mean of 10 counts.

Table 1

Experiments of June 6, 1936. Lutescens 0329 Wheat. Duration of Yarovization, 50 Days

Amount of bios preparation added in cm^3	Number of yeast cells in chamber division when bios preparation is added		Ratio of yarovized to unyarovized
	from unyarovized embryos	from yarovized embryos	
0.2	4.2	13.0	3.1
0.5	12.8	49.7	3.9
0.75	25.9	80.9	3.1

Table 2

Experiment of June 8, 1936. Kooperatorka Wheat. Duration of Yarovization 50 Days

Amount of bios preparation added in cm^3	Number of yeast cells in chamber division when bios preparation is added		Ratio of yarovized to unyarovized
	from unyarovized embryos	from yarovized embryos	
0.2	1.8	4.7	2.6
0.5	1.6	4.2	2.6
1.0	2.5	7.5	3.0
1.5	2.6	7.5	3.0

A comparison of the bios content in yarovized and unyarovized embryos shows that it is considerably greater in yarovized embryos, the ratio of yarovized to unyarovized in some cases exceeding 3.

The experiments of June 9 and 23 show that even a short exposure to low temperatures causes a marked rise in the production of bios in young embryos, as compared with the controls.

Table 3

Experiments of June 9, 1936. Kooperatorka Wheat. Duration of Yarovization, 10, 20, 30, 40 and 50 days. Amount of Bios Preparation Added in All the Tests, 0.5 cm³

Duration of yarovization	Number of yeast cells in chamber division when bios preparation is added	Ratio of yarovized to unyarovized
Five-day-old seedlings from unyarovized seeds	10.0	—
Yarovized for 10 days	22.5	2.25
» » 20 »	25.0	2.5
» » 30 »	22.7	2.27
» » 40 »	19.1	1.91
» » 50 »	19.8	1.98

Table 4

Experiments of June 23, 1936. Kooperatorka Wheat. Duration of Yarovization, 10, 20, 30, 40 and 50 Days. Amount of Bios Preparation Added in All the Tests, 0.5 cm³

Duration of yarovization	Number of yeast cells in chamber division when bios preparation is added	Ratio of yarovized to unyarovized
Five-day-old seedlings from unyarovized seeds	2.4	—
Yarovized for 10 days	9.6	4.0
» » 20 »	12.9	5.3
» » 30 »	13.1	5.4
» » 40 »	21.0	8.7
» » 50 »	17.2	7.1

The above-cited data indicate that, during the process of yarovization, an increased formation of substances of bios takes place in the young seed embryos of winter wheat. This justifies one in believing these substances to participate processes, which cause an acceleration in the development of plants to take place.

Many investigations have been devoted to the elucidation of the chemical nature of bios. E. Wildiers([16]), Dagys([10], [11]) and Gromakovskij([14]) are inclined to regard bios as a constituent element of the protein molecule.

Sakamura and Janagihara([15]) consider the hormones to be the result of a proteolytic splitting of protein. Sukhorukov([8]) confirms experimentally the possibility of the partial adsorption of bios by protein and by the first products of its disintegration.

In our paper on the physiological characteristics of yarovized winter wheats([13]), we noted the comparatively large mobility of the protein complex in the young embryo of yarovized seeds. We also noted its lower thermostability, greater permeability of leaf tissue, greater number of free NH_2 groups. The great mobility of the protein complex and the high bios content typical for yarovized wheats, are, in the light of the above-cited

literature, a new confirmation of the genetic relationship existing between the formation of bios in the cells and the structural state of the protein.

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ERRATA

In the N. Ermolaev's paper (1937, vol. XVI, № 4) read:

1) Formula (3):

$$V^{\rho}_{k,m}(x) = \frac{2x^{2k+\rho}}{\sqrt{\pi^3}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{\infty}\int_0^{\infty} e^{-(u+v)}\, u^{\nu-\frac{1}{2}}\, v^{k-\rho-\frac{3}{2}} \left(1+\frac{uv}{x^2}\right)^{n-\frac{1}{2}} du\, dv, \qquad (3)$$

2) The second line of formula (10):

$$+\frac{\omega^2}{1+\omega^2}\left[\frac{2\,(2\omega^2-1)}{3\,(1+\omega^2)}+k+\rho+2\sigma+n-\nu+\frac{7}{6}\right]. \qquad (10)$$

3) The second line of formula (12):

$$+\frac{\omega^2}{1+\omega^2}\left[\frac{2\,(2\omega^2-1)}{3\,(1+\omega^2)}+2\sigma+k+\rho+n-\nu+\frac{7}{6}\right]. \qquad (12)$$

4) The last formula

$$c' = \frac{2\omega^2}{1+\omega^2}\left[\frac{2\omega^2-1}{3\,(1+\omega^2)}+\sigma+\frac{1}{12}\right]+\sigma^2+\nu^2-\nu\sigma-\sigma-\frac{5}{24}.$$

In the S. Bellustin's paper (1937, vol. XVI, № 6) substitute:

1) Equation (1) and on p. 300, 301, 302:

Φ for φ.

2) Equation (14′)

$15s^2$ for $45s^2$.

In the J. Larionov's and A. Seidel's paper (1937, vol. XVI, № 9) substitute:

1) Page 444:

$^3S_{\frac{1}{2}}$ for $^3S_{\frac{1}{2}}$.

2) Page 445:

10^{-6} for 10^{-8}.